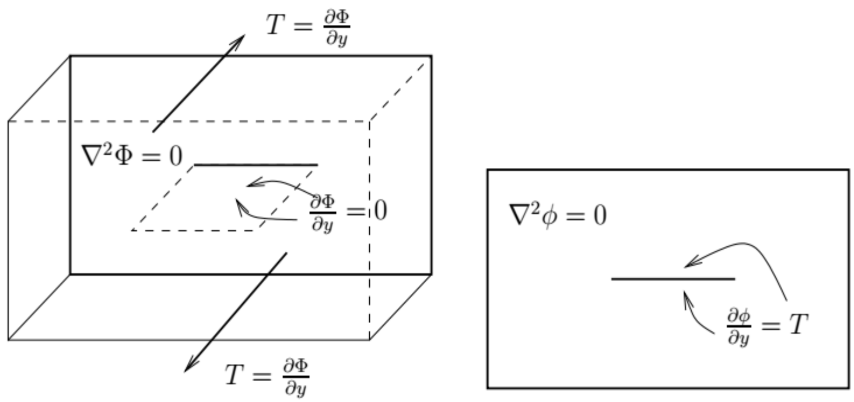
Lecture 11: Fracture in solids

A famous problem in elasticity is to calculate the displacement field $(0, 0, \Phi(x, y))$ in antiplane strain around a crack at $y = 0$, $0 < x < c$, as illustrated in the left-hand figure below. The displacement Φ is such that $\nabla^2\Phi = 0$ except on $y = 0$, $0 \le x \le c$; $\lim_{y\downarrow\uparrow 0} \partial\Phi/\partial y = 0$ for $0 < x < c$ (zero traction on the crack surface); $|\nabla\Phi|$ has an inverse square-root singularity at $(0, 0)$ and at $(c, 0)$ (so that the displacement Φ is finite at the crack tips); $\partial\Phi/\partial y = T + \mathrm{O}(r^{-2})$ as $r^2 = x^2 + y^2 \to \infty$ (uniform shearing at large distances). Setting $\Phi = Ty - \phi(x, y)$ and $\phi_y = Im(w(z))$, we find that $w(z)$ is holomorphic on $\mathbb{C}\backslash\overline{\Gamma}$; $Im(w_\pm) = T$ on Γ; $w = O(z^{-1/2})$ as $z \to 0$ and $w = O((z - c)^{-1/2})$ as $z \to c$; $w = O(z^{-2})$ as $z \to \infty$.



By problem 2, case (2) in the lecture 11, a solution for $w(z)$ is given by

$$w(z) = \tilde{w}(z)\left[\frac{1}{\pi}\int_0^c \frac{g(\xi)\,\mathrm{d}\xi}{\tilde{w}_+(\xi)(\xi - z)} + \tilde{H}(z)\right],$$

where $g = T$ and $\tilde{H}(z)$ is an arbitrary function of z holomorphic on $\mathbb{C}\backslash\{0, c\}$. Choosing $\tilde{w} = z^{-1/2}(c - z)^{-1/2}$ on the plane cut along Γ, with $\tilde{w}_\pm(x) = \pm x^{-1/2}(c - x)^{-1/2}$ for $0 < x < c$, gives

$$w(z) = \frac{1}{z^{1/2}(c - z)^{1/2}}\left[\frac{1}{\pi}\int_0^c \frac{\xi^{1/2}(c - \xi)^{1/2}g(\xi)}{(\xi - z)}\mathrm{d}\xi + \tilde{H}(z)\right]. \qquad (17)$$

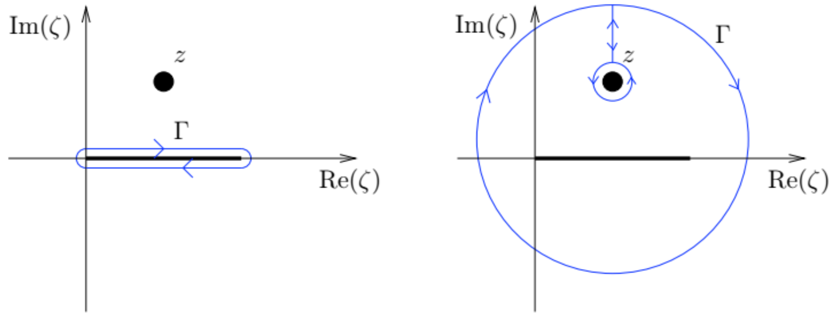
At the endpoints of Γ, the integral in (17) is finite (because of the choice we made for $\tilde{w}(z)$) and $\tilde{H}(z)$ can only have isolated singularities (because $\tilde{H}(z)$ is holomorphic on $\mathbb{C}\backslash\{0, c\}$). Since $w = O(z^{-1/2})$ as $z \to 0$ and $w = O((c - z)^{-1/2})$ as $z \to c$, $\tilde{H}(z)$ must have removable singularities at $z = 0$ and $z = c$, and is therefore entire (*i.e.* holomorphic on $\mathbb{C}$). Moveover, $w = O(z^{-2})$ as $z \to \infty$ if and only if $\tilde{H}(z) = O(z^{-1})$ as $z \to \infty$, so $\tilde{H}(z) = 0$ by Liouville's theorem. Hence, the solution is given by

$$w = \frac{T}{\pi z^{1/2}(c - z)^{1/2}}\int_0^c \frac{\xi^{1/2}(c - \xi)^{1/2}\,\mathrm{d}\xi}{(\xi - z)}.$$

Amazingly, this integral can be evaluated explicitly. First note that

$$\int_0^c \frac{\xi^{1/2}(c - \xi)^{1/2}\,\mathrm{d}\xi}{(\xi - z)} = \frac{1}{2}\int_\Gamma \frac{\zeta^{1/2}(c - \zeta)^{1/2}\,\mathrm{d}\zeta}{(\zeta - z)}$$

because the square root changes sign on the bottom integral, while the direction of integration is reversed.



Now deform the contour to infinity. There is a residue contribution from the pole at $\zeta = z$ of

$$\frac{2\pi \mathrm{i}}{2} z^{1/2}(c-z)^{1/2}.$$

To evaluate the contribution from a large circle at infinity expand the integrand as

$$\frac{\zeta^{1/2}(c-\zeta)^{1/2}}{(\zeta - z)} \sim -\mathrm{i}\left(1-\frac{c}{\zeta}\right)^{1/2}\left(1-\frac{z}{\zeta}\right)^{-1} \sim -\mathrm{i}\left(1+\frac{2z-c}{2\zeta}+\cdots\right) \tag{18}$$

which integrates to $-\pi(z - c/2)$. Thus

$$w = \frac{T}{\pi z^{1/2}(c-z)^{1/2}}\left(\pi \mathrm{i} z^{1/2}(c-z)^{1/2} - \pi\left(z-\frac{c}{2}\right)\right) = T\mathrm{i} - \frac{T(z-c/2)}{z^{1/2}(c-z)^{1/2}}.$$

Example: Aerodynamics

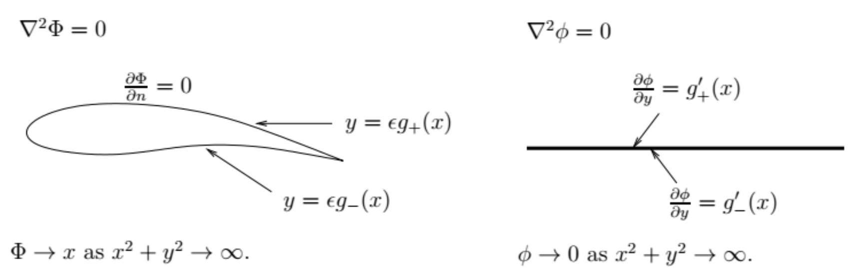
Here the physical model is the flow of a uniform stream of ideal fluid past a thin aerofoil with a blunt nose, a sharp trailing and a small angle of attack, as illustrated in the figure below. We denote the boundary of the aerofoil by $y = \varepsilon g_{\pm}(x)$ for $0 < x < c$, where $g_- \le g_+$ and $\varepsilon \ll 1$. If $\Phi(x,y)$ is the velocity potential, then $\nabla^2\Phi = 0$ in the fluid surrounding the aerofoil. The no-flux boundary condition states that $\partial\Phi/\partial n = 0$ on the boundary of the aerofoil, except at the sharp trailing edge where the outward normal derivative is not defined and the fluid velocity is finite according to the Kutta-Joukowski condition. Finally, $|\nabla\Phi - (1,0)| = \mathrm{O}(r^{-1})$ as $r^2 = x^2 + y^2 \to \infty$. Expanding $\Phi \sim x + \varepsilon\phi$ gives $\nabla^2\phi = 0$ in the fluid. Since the outward normal to the upper surface of the aerofoil is $(-\varepsilon g_+', 1)$, the no-flux boundary condition on the upper surface implies

$$\begin{aligned} 0 &= (-\varepsilon g_+', 1)\cdot\nabla\Phi \\ &= (1,0)\cdot(-\varepsilon g_+', 1) + \varepsilon(\phi_x(x,\varepsilon g_+), \phi_y(x,\varepsilon g_+))\cdot(-\varepsilon g_+', 1) \\ &= -\varepsilon g_+' + \varepsilon\phi_y(x,0) + O(\varepsilon^2) \end{aligned}$$

as $\varepsilon \to 0$. A similar expansion holds for the no-flux boundary condition on the lower surface, giving, at leading order, the boundary conditions

$$\phi_y(x, \pm 0) = g_{\pm}'(x) \quad \text{for} \quad 0 < x < c.$$

Thus, at leading order the wing is replaced by boundary conditions on $y = \pm 0$, $0 < x < c$. Fluid dynamics also tells us the behaviour of the velocity near the endpoints of the wing: $|\nabla\phi| = O(r^{-1/2})$ as $r \to 0$ and $|\nabla\phi| = O(1)$ as $(x,y) \to (c,0)$. Finally, in the far field, $|\nabla\phi| = O(r^{-1})$ as $r \to \infty$.



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Let $w = -(\phi_x - \mathrm{i}\phi_y)$, then $w(z)$ is holomorphic on $\mathbb{C}\backslash\overline{\Gamma}$, $Im(w_\pm) = g'_\pm$ on Γ, $w = O(z^{-1/2})$ as $z \to 0$, $w = O(1)$ as $z \to c$ and $w = O(z^{-1})$ as $z \to \infty$.

For a symmetric wing, $g_+ = -g_-$, so $g'_+ = g'_-$ and we must solve an easy problem as in problem 2, case (1) of lecture 11. A zero-thickness aerofoil has $g_+ = g_-$, so $g'_+ = g'_-$ and we must solve a harder problem as in problem 2, case (2) of lecture 11.

$g_+ = -g_-$ $\qquad\qquad$ $g_+ = g_-$

In the second case, we let $g'_+ = g'_- = g$ and choose $\tilde{w} = z^{-1/2}(c-z)^{-1/2}$ so that we can use the same solution as for the crack problem, namely (17). As in the crack problem, $\tilde{H}(z)$ can only have isolated singularities at the endpoints of Γ and is therefore entire. However, $w = O(z^{-1})$ as $z \to \infty$ if and only if $\tilde{H}(z) = O(1)$ as $z \to \infty$, so $\tilde{H}(z)$ is constant by Liouville's theorem (in contrast to the crack problem). Finally, we ensure that w is finite as the trailing edge $z = c$ by setting

$$\tilde{H}(z) = \tilde{H}(c) = -\frac{1}{\pi}\int_0^c \frac{g(\xi)\xi^{1/2}(c-\xi)^{1/2}}{\xi - z}\mathrm{d}\xi\bigg|_{z=c},$$

giving

$$\begin{aligned} w(z) &= \frac{1}{\pi z^{1/2}(c-z)^{1/2}}\int_0^c g(\xi)\xi^{1/2}(c-\xi)^{1/2}\left(\frac{1}{\xi-z} - \frac{1}{\xi-c}\right)\mathrm{d}\xi \\ &= \frac{(c-z)^{1/2}}{\pi z^{1/2}}\int_0^c \frac{g(\xi)\xi^{1/2}}{(c-\xi)^{1/2}(\xi-z)}\,\mathrm{d}\xi. \end{aligned}$$

It is an exercise in perturbation methods (see C6.3a) to verify that $w(z) = \mathrm{O}(1)$ as $z \to c$. The solution shows that the behaviour of w near the end points may be built into the particular solution by taking $\tilde{w}(z) = (c-z)^{1/2}/z^{1/2}$.

General Hilbert problem

We have seen that when $w_+ - w_-$ is given on Γ we can solve immediately for f and therefore for w. When $w_+ + w_-$ is given on Γ we find a singular integral equation for f, but we can find w (and f) by introducing $\tilde{w}$ such that $\tilde{w}_+ = -\tilde{w}_- \neq 0$ on Γ. What about more general relations between w_+ and w_- on Γ? The general Hilbert problem is

$$a(z)w_+ + b(z)w_- = c(z) \quad \text{on} \quad \Gamma,$$

with a, $b \neq 0$ and c prescribed on Γ. Suppose we can find $\tilde{w}$ holomorphic and non-zero away from Γ, with

$$a(z)\tilde{w}_+ = -b(z)\tilde{w}_- \neq 0 \quad \text{on} \quad \Gamma.$$

Let $W = w/\tilde{w}$ then

$$W_+ - W_- = \frac{w_+}{\tilde{w}_+} - \frac{w_-}{\tilde{w}_-} = \frac{w_+}{\tilde{w}_+} - \frac{w_-}{-a\tilde{w}_+/b} = \frac{aw_+ + bw_-}{a\tilde{w}_+} = \frac{c}{a\tilde{w}_+} \quad \text{on} \quad \Gamma,$$

giving

$$W(z) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{c(\zeta)}{a(\zeta)\tilde{w}_+(\zeta)(\zeta - z)}\mathrm{d}\zeta + H(z),$$

where $H(z)$ is an arbitrary function of z that is holomorphic away from the endpoints of Γ. For $\tilde{w}$ we have $\tilde{w}_+/\tilde{w}_- = -b/a$, so $\log(\tilde{w}_+) - \log(\tilde{w}_-) = \log(-b/a)$ on Γ. We can therefore use the same method as before to solve for $\tilde{w}$ and hence find w.

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Example: The Cauchy singular integral equation for f,

$$a(z)f(z) + b(z)\int_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} = c(z), \tag{19}$$

can be rewritten as a Hilbert problem for

$$w(z) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{f(z)}{\zeta - z}\mathrm{d}\zeta,$$

using the Plemelj formulae. Hence we can solve (19) explicitly (see sheet 5).