Lecture 14: The Wiener-Hopf method

The Wiener-Hopf method

Recommended reading: Functions of a complex variable, theory and technique by Carrier, Krook and Pearson.

Example: An ODE problem

Suppose the complex-valued function $y(x)$ satisfies

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a^2y = 0 \text{ for } x > 0, \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b^2y = 0 \text{ for } x < 0,$$

with

$$y(0+) - y(0-) = 0, \quad \frac{\mathrm{d}y}{\mathrm{d}x}(0+) - \frac{\mathrm{d}y}{\mathrm{d}x}(0-) = 1,$$

and $y(x) \to 0$ as $|x| \to \infty$, where $a,\ b \in \mathbb{C}$ with $a \neq b$ and $Im(a),\ Im(b) > 0$.

The solution may be readily found via elementary methods to be given by

$$y(x) = \begin{cases} \dfrac{\mathrm{e}^{-\mathrm{i}bx}}{\mathrm{i}(a+b)} & \text{for } x < 0, \\ \dfrac{\mathrm{e}^{\mathrm{i}ax}}{\mathrm{i}(a+b)} & \text{for } x > 0, \end{cases}$$

so that

$$y(x) = \begin{cases} \mathrm{O}\left(\mathrm{e}^{Im(b)x}\right) & \text{as } x \to -\infty, \\ \mathrm{O}\left(\mathrm{e}^{-Im(a)x}\right) & \text{as } x \to \infty, \end{cases} \tag{26}$$

That the solution decays exponentially at infinity allows the problem to be solved using the Wiener-Hopf method. We shall therefore use this simple ODE problem to introduce the Wiener-Hopf method, before moving onto more complicated integral equation and mixed-boundary-value problems.

We begin by assuming that $y(x) = \mathrm{O}(\mathrm{e}^{\alpha x})$ as $x \to \infty$ and $y(x) = \mathrm{O}(\mathrm{e}^{\beta x})$ as $x \to -\infty$ for some real constants $\alpha < \beta$ whose existence we shall need to verify *a posteriori* (*i.e.* having found the solution). We then define the "half-functions"

$$y_+(x) = \begin{cases} 0 & \text{for } x < 0, \\ y(x) & \text{for } x > 0, \end{cases} \quad y_-(x) = \begin{cases} y(x) & \text{for } x < 0, \\ 0 & \text{for } x > 0. \end{cases}$$

so that

$$\bar{y}_+(k) = \int_{-\infty}^{\infty} y_+(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = \int_0^{\infty} y(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x$$

is holomorphic in $Im(k) > \alpha$ (because $y_+(x)$ is piecewise continuous and $y_+(x) = \mathrm{O}(\mathrm{e}^{\alpha x})$ as $x \to \infty$) and

$$\bar{y}_-(k) = \int_{-\infty}^{\infty} y_-(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = \int_{-\infty}^{0} y(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x$$

is holomorphic in $Im(k) < \beta$ (because $y_-(x)$ is piecewise continuous and $y_-(x) = \mathrm{O}(\mathrm{e}^{\beta x})$ as $x \to -\infty$).

Now,

$$\begin{aligned} 0 &= \int_0^{\infty} (y''(x) + a^2y(x))\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x \\ &= \left[y'(x)\mathrm{e}^{\mathrm{i}kx}\right]_{x=0+}^{x=\infty} - \int_0^{\infty} y'(x)\mathrm{i}k\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x + a^2\bar{y}_+(k) \\ &= \left[(y'(x) - \mathrm{i}ky(x))\mathrm{e}^{\mathrm{i}kx}\right]_{x=0+}^{x=\infty} + \int_0^{\infty} y(x)(\mathrm{i}k)^2\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x + a^2\bar{y}_+(k) \\ &= -y'(0+) + \mathrm{i}ky(0+) + (a^2 - k^2)\bar{y}_+(k) \end{aligned}$$

provided $\bar{y}_+(k)$ exists and $(y'(x) - \mathrm{i}ky(x))\mathrm{e}^{\mathrm{i}kx} \to 0$ as $x \to \infty$, which is the case for $Im(k) > \alpha$. Similalry,

$$0 = y'(0-) - \mathrm{i}ky(0-) + (b^2 - k^2)\bar{y}_-(k)$$

provided $Im(k) < \beta$. Using the jump conditions at $x = 0$ to eliminate the unknowns $y(0\pm)$ and $y'(0\pm)$ gives

$$(k^2 - a^2)\bar{y}_+(k) + (k^2 - a^2)\bar{y}_-(k) = \left[\mathrm{i}ky(x) - y'(x)\right]_{x=0-}^{x=0+} = -1, \tag{27}$$

for $\alpha < Im(k) < \beta$. This is one equation relating the two unknowns $\bar{y}_-(k)$ and $\bar{y}_+(k)$. However, there exists enough auxiliary information to determine them using the Wiener-Hopf method, as follows.

Provided $k \neq a, -b$ we can divide (27) through by $(k-a)(k+b)$ and rearrange to obtain

$$\left(\frac{k+a}{k+b}\right)\bar{y}_+(k) + \left(\frac{k-b}{k-a}\right)\bar{y}_-(k) = -\frac{1}{(k-a)(k+b)} = \frac{1}{a+b}\left(\frac{1}{k+b} - \frac{1}{k-a}\right)$$

so that

$$\left(\frac{k+a}{k+b}\right)\bar{y}_+(k) - \frac{1}{(a+b)(k+b)} = -\left(\frac{k-b}{k-a}\right)\bar{y}_-(k) - \frac{1}{(a+b)(k-a)}. \tag{28}$$

The left-hand side of (28) is holomorphic in $Im(k) > \alpha_1 := \max(\alpha, -Im(b))$, while the right-hand side is holomorphic in $Im(k) < \beta_1 = \min(\beta, Im(a))$. Since $\alpha < \beta$ by assumption and $-Im(b) < 0 < Im(a)$, it follows that $\alpha_1 < \beta_1$ so that these half-planes overlap in the "overlap strip" $\alpha_1 < Im(k) < \beta_1$. Hence, the left- and right-hand sides of (28) are equal on a dense set of points, and therefore the right-hand side is the analytic continuation of the left-hand side into the lower half-plane, so together they define an entire function, $E(k)$ say. Since $\bar{y}_+(k) \to 0$ as $k \to \infty$ in $Im(z) > \alpha_1$ and $\bar{y}_-(k) \to 0$ as $k \to \infty$ in $Im(z) < \beta_1$, we deduce that $E(k) \to 0$ as $k \to \infty$. By Liouville's theorem, we deduce that $E(k) = 0$, giving

$$\bar{y}_+(k) = \frac{1}{(a+b)(k+a)}, \quad \bar{y}_-(k) = -\frac{1}{(a+b)(k-b)},$$

which may be inverted to obtain the exact solution given above (taking care to close the contour at $\mp\mathrm{i}\infty$ as appropriate).

The Wiener-Hopf method

The canonical Wiener-Hopf problem is to find $\overline{u}_+ \in H(Im(k) > \alpha)$ and $\overline{v}_- \in H(Im(k) < \beta)$ when

$$F(k)\overline{u}_+(k) + \overline{v}_-(k) = G(k) \quad \text{on } \Omega = \{k : \alpha < Im(k) < \beta\},$$

where $F, G \in H(\Omega)$ are prescribed and $F \neq 0$ on Ω.

Note that it is conventional to use subscripts $_+(_-)$ to indicate that a function is holomorphic in some upper (lower) half-plane, *e.g.* $(k-\mathrm{i})^{-1} = (k-\mathrm{i})_+^{-1} = (k-\mathrm{i})_-^{-1}$. Be careful not to confuse this notation with the subscripts $_\pm$ on a function of a real variable for which $\psi_\pm(x) := \mathcal{H}(\pm x)\psi(x)$ (whose Fourier transform is a $\pm$ function only if $\psi = \mathrm{O}(\mathrm{e}^{c_\pm|x|})$ as $x \to \pm\infty$ for some real constant $c_\pm$).

The Wiener-Hopf method for solving this problem is as follows.

1. **Factorization of F:** Find $M_+ \in H(Im(k) > \alpha_1)$ and $N_- \in H(Im(k) < \beta_1)$ such that

$$F = \frac{M_+}{N_-} \qquad \text{on } \Omega_1 = \{k : \alpha_1 < Im(k) < \beta_1\} \subseteq \Omega.$$

   Note that $M_+ \neq 0$, $N_- \neq 0$ on Ω_1 by cancelling common factors. It follows that

$$M_+\overline{u}_+ + N_-\overline{v}_- = N_-G \quad \text{on } \Omega_1.$$

2. **Factorization of N_-G:** Find $P_+ \in H(Im(k) > \alpha_2)$ and $Q_- \in H(Im(k) < \beta_2)$ such that

$$N_-G = P_+ + Q_- \quad \text{on } \Omega_2 = \{k : \alpha_2 < Im(k) < \beta_2\} \subseteq \Omega_1.$$

   It follows that

$$M_+\overline{u}_+ + N_-\overline{v}_- = P_+ + Q_- \quad \text{on } \Omega_2.$$

3. **Analytic continuation:** Define $E = M_+\overline{u}_+ - P_+ = Q_- - N_-\overline{v}_-$ on Ω_2. Since $\alpha_2 < \beta_2$, we can analytically continue E into $\mathbb{C}\backslash\Omega_2$ and use the identity theorem (E and $M_+\overline{u}_+ - P_+$ and $Q_- - N_-\overline{v}_-$ are identical on the dense set Ω_2) to deduce that

$$E = \begin{cases} M_+\overline{u}_+ - P_+, & Im(k) > \alpha_2, \\ Q_- - N_-\overline{v}_-, & Im(k) < \beta_2, \end{cases}$$

Thus, E is entire, *i.e.* $E \in H(\mathbb{C})$.

4. **Behaviour at infinity:** Since M_+, P_+, N_-, and Q_- are known functions of k, we know their behaviour as $|k| \to \infty$. The behaviour of $\overline{u}_+$ in $Im(k) > \alpha_2$ and of $\overline{v}_-$ in $Im(k) < \beta_2$ as $|k| \to \infty$ is determined by the behaviour of u_+, v_- as $x \to \pm 0$ via an asymptotic expansion of the relevant Fourier integral (this is usually given in questions). This determines the behaviour of $E(k)$ as $|k| \to \infty$, and we can then apply Liouville's theorem as follows.

   - If $E \to$ constant as $|k| \to \infty$, E is entire and bounded, and therefore a constant by Liouville's theorem; thus, if $E \to 0$ as $|k| \to \infty$, then $E = 0$.
   - If $E = \mathrm{O}(k^n)$ as $|k| \to \infty$, where $n \in \mathbb{N}$, then $\mathrm{d}^n E/\mathrm{d}k^n$ is entire and bounded, and therefore a constant by Liouville's theorem, so E is a polynomial of degree n.

   To solve the system for $n \geq 0$, we therefore need to determine $n+1$ coefficients by analysing further the boundary conditions. This means that in practice we want n to be as small as possible.

5. **Invert:** To find u_+ and v_- we need to invert

$$\overline{u}_+ = \frac{P_+ + E}{M_+}, \qquad \overline{v}_- = \frac{Q_- - E}{N_-}.$$

   Note that the inversion contours must be in Ω_2.

emarks

- Steps 1 and 2 in the Wiener-Hopf method are called Wiener-Hopf factorizations or decompositions (product and sum, respectively). These are not unique, *e.g.* we can multiply M_+ and N_- by any entire function, but then E has worse behaviour at infinity.
- In many applications we can spot the factorizations, though we will describe a constructive method below.