

# Lecture 3: Revision of conformal mapping

We now look at holomorphic functions from the geometric point of view, as well as analytically. That is, we see what the image of a given domain is when we apply a holomorphic function, thought of as a mapping, to it. Then, we look at some applications. In particular, we combine two ideas:

- The composition of holomorphic functions is holomorphic;
- The real and imaginary parts of a holomorphic function are solutions of Laplace's equation.

This means that if we know a solution of Laplace's equation in one domain, usually a simple one, we can transform it into a solution in another domain by a holomorphic mapping. If the target domain is the (perhaps complicated) domain in which we want to solve Laplace's equation, and if we can fix up the boundary conditions, we have the solution to our problem.

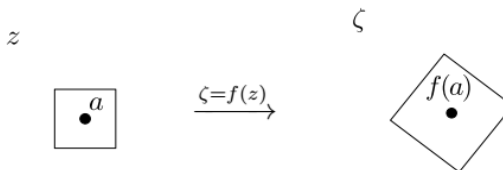
We begin by reviewing some properties of conformal mapping. As a matter of terminology, we usually write mappings as  $\zeta = f(z)$ ; the words 'horizontal' and 'vertical' mean parallel to the real axis and imaginary axis respectively. (If you are reading this at an angle, tough.)

## Conformal maps

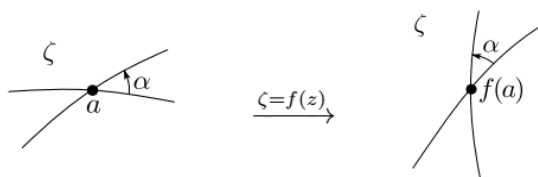
Suppose  $f(z)$  is holomorphic at  $z = a$ . Then, by Taylor's theorem, for  $z$  near  $a$  we have

$$\zeta = f(z) = f(a) + f'(a)(z - a) + \dots$$

If  $f'(a) \neq 0$ , this shows that a small neighbourhood of  $z = a$  is simply translated, with  $z = a$  going to the point  $\zeta = f(a)$ , and then rotated and scaled by being multiplied by  $f'(a)$  as illustrated. The map is clearly locally one-to-one, being a small perturbation of a linear map.

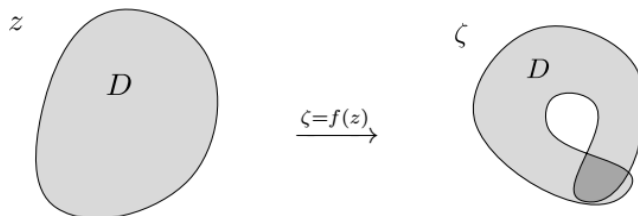


If two curves meet at  $z = a$  and the angle between them is  $\alpha$ , it follows from this local linearity that the angle between their images is also  $\alpha$  (and has the same sense).



Maps with this property are called *conformal*, and we have showed that a holomorphic function is a conformal map at all points where its derivative does not vanish.

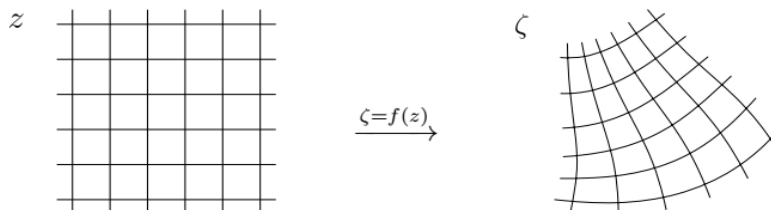
This is only a local statement. That is, when we look at the image of a domain  $D$  under  $f(z)$ , that even if  $f'(z)$  does not vanish in  $D$ , the map may still not be globally one-to-one.



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It is evident that the composition of two conformal maps is itself conformal. We shall use this in building up complicated maps from simple ones.

Lastly note that it follows from conformality that the image of an orthogonal (Cartesian) coordinate grid is a set of orthogonal curvilinear coordinates. All two-dimensional orthogonal coordinate systems can be generated in this way.

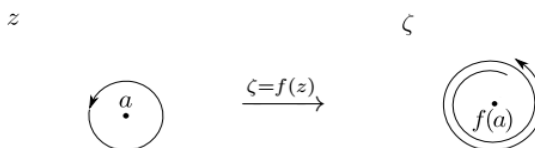


## Behaviour near a critical point

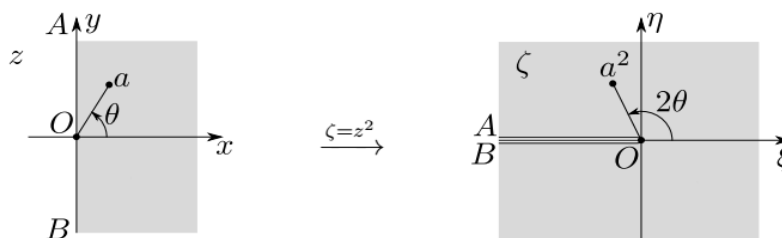
Now suppose that  $f'(a) = 0$  and  $f''(a) \neq 0$ . If we write  $z = a + re^{i\theta}$ , then

$$\begin{aligned} f(z) &= f(a) + \frac{1}{2}f''(a)(z - a)^2 + O((z - a)^3) \\ &= f(a) + \frac{1}{2}f''(a)r^2e^{2i\theta} + O(r^3). \end{aligned}$$

The image of a small neighbourhood of  $z = a$  now covers a small neighbourhood of  $f(a)$  *twice*, and the angle between two curves meeting at  $a$  is doubled ( $\theta$  above is the angle between two particular curves, a line parallel to the  $x$  axis and a line at an angle  $\theta$  to the axis, but the doubling of angles clearly holds in general).



We see that if a map  $f(z)$  has a critical point at a point  $z = a$  in a region  $D$ , it is neither conformal at that point, nor locally one-to-one near it. The only hope of constructing a one-to-one map using  $f(z)$  is if  $a$  lies on the boundary of  $D$ . A very simple example is  $f(z) = z^2$  acting on the right-hand half-plane  $x > 0$ . The image of an interior point  $z = re^{i\theta}$ , where  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ , is  $\zeta = r^2e^{2i\theta}$ , and we see that  $-\pi < \arg \zeta < \pi$ , so the map is one-to-one (as predicted by the doubling of the angles).



For more general domains  $D$  and maps  $f(z)$  this suggests that, if  $\partial D$  has a smooth tangent at  $z = a$ , and  $f'(a) = 0$  but  $f''(a) \neq 0$ , the map has a chance of being locally one-to-one and mapping a neighbourhood of  $z = a$  to a neighbourhood of a cusp in the boundary of the image domain. Whether this occurs depends rather delicately on the details of the curvature of  $\partial D$  at  $z = a$  and on the higher derivatives of  $f(z)$  at  $z = a$ . To

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illustrate how these may be analysed, let us take  $a = 0$ ,  $f(a) = 0$ ,  $D$  to be the right-hand half-plane  $x > 0$ , and suppose that, near  $z = 0$ ,

$$f(z) \sim z^2 + cz^3 + O(z^4)$$

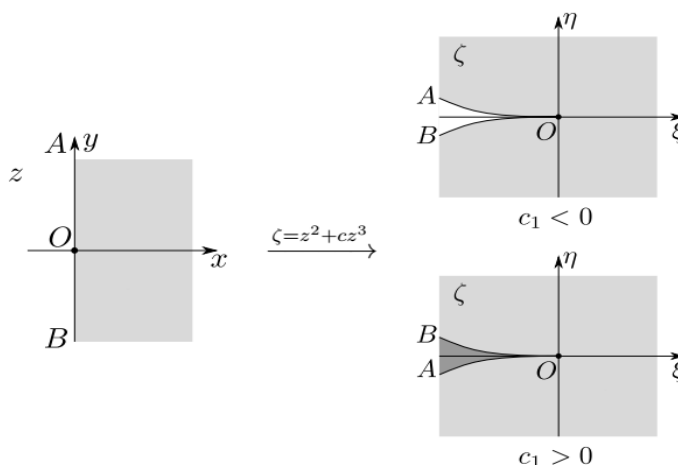
where  $c = c_1 + ic_2 \neq 0$ . We are going to do a local analysis of the image domain: it says nothing about global behaviour. The image of a point  $z = iy$  on the imaginary axis (ie on  $\partial D$ ) is

$$\zeta = \xi + i\eta = (iy)^2 + (c_1 + ic_2)(iy)^3 + O(y^4)$$

so, equating real and imaginary parts, the image curve is locally given parametrically by

$$\xi = -y^2 + O(y^3), \quad \eta = -c_1y^3 + O(y^4).$$

When  $y > 0$  this curve must have  $\eta > 0$  (and vice-versa for  $y < 0$ ) for the image to be one-to-one. Hence the map is locally one-to-one if  $c_1 < 0$ , but not if  $c_1 > 0$ . (The case  $c_1 = 0$  can only be decided by going to higher order terms.)



## The Riemann mapping theorem

It is natural to ask what domains we can map to what. The answer, due to Riemann, is still astonishing one and a half centuries after it was proved.

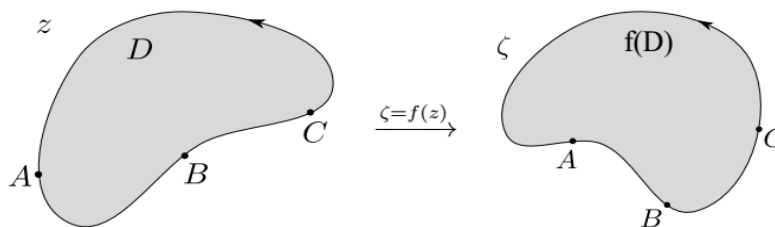
**The Riemann mapping theorem.** *Any simply connected domain  $D$ , with the sole exception of  $\mathbb{C}$  itself, can be mapped conformally onto the unit disc  $|\zeta| < 1$ . There are three free real parameters in the map; if  $D$  is bounded by a Jordan curve (which may pass through the point at infinity), we may specify the images of three boundary points and the map is then uniquely determined. Alternatively, we may specify the image of one interior point of  $D$ , and the image of a direction at that point.*

The proof is very technical; unfortunately it is not constructive (ie it does not give a recipe for constructing the map). Hence we omit it, just making some comments on the theorem.

1. The reason that there is no map from  $\mathbb{C}$  itself to the unit disc is that if there were one, the mapping function would be entire and its modulus would be bounded by 1; by Liouville's theorem, the function would be constant (and hence not conformal).
2. The three degrees of freedom look mysterious, but as we shall see there is precisely a three-parameter family of maps from the unit disc to itself, so having arrived there one can apply any of those maps to fix the boundary points.

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3. It is convenient (and often conventional) to map to the unit disc. However, any other ‘canonical domain’ would do, since we can map the disc onto it and use the composition of maps mentioned above. The upper half-plane is also often used as a canonical domain, and then the three boundary points in Riemann’s theorem are usually taken to be 0, 1 and  $\infty$ .
4. The theorem says that the map is conformal in the interior of  $D$  (so its inverse is conformal in  $|\zeta| < 1$ ), but it says nothing about the behaviour on the boundary, which can be extremely complicated (singularities being required to smooth out corners and cusps; see §2.3). Fortunately such difficulties rarely arise in practice.
5. Because conformal maps preserve angles, including their direction, if we go round  $\partial D$  in a particular direction (eg anticlockwise), the image of  $\partial D$  is traversed in the same sense. This is often useful in calculating the effect of a specific map; see the examples below.



### Standard maps

Let us look at some standard maps. In the accompanying examples, we see how to construct complicated maps from these building blocks.

#### Bilinear maps (Möbius transformations)

The Möbius transformation is

$$\zeta = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C},$$

where  $ad - bc \neq 0$  (otherwise, the map is constant) and, to avoid trivial translations, rotations and scalings,  $c \neq 0$ . As

$$\frac{az + b}{cz + d} = \frac{a}{c} + \frac{bc - ad}{c} \frac{1}{cz + d},$$

the map is a composition of translations, rotations and scalings, plus one *inversion* with respect to the point  $z = -d/c$ . That is, we can write it as the chain

$$\begin{aligned} \zeta_1 &= cz + d && \text{(translation and rotation/scaling),} \\ \zeta_2 &= \frac{1}{\zeta_1} && \text{(inversion),} \\ \zeta &= \frac{a}{c} + \frac{bc - ad}{c} \zeta_2 && \text{(translation and rotation/scaling).} \end{aligned}$$

The only nontrivial part is the inversion.

It is a well known fact that Möbius transformations take circles and straight lines (which are circles with their centre at infinity) into circles and straight lines. This is trivial for translations, rotations and scalings, so we need only show it for an inversion. Let us therefore compute the image of the general circle

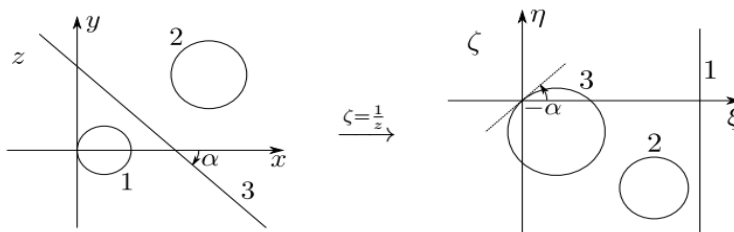
$$\alpha^2 z \bar{z} + \bar{\beta} z + \beta \bar{z} + \gamma^2 = 0,$$

where  $\alpha$  and  $\gamma$  are real (if  $\alpha = 0$  the circle is a line) under the map  $\zeta = 1/z$  (inversion with respect to the origin). By direct substitution, it is

$$\alpha^2 + \bar{\beta} \bar{\zeta} + \beta \zeta + \gamma^2 \zeta \bar{\zeta} = 0,$$

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and this is clearly a circle by comparison with the equation for the original one ( $\alpha \longleftrightarrow \gamma, \beta \longleftrightarrow \bar{\beta}$ ). If the original circle passes through the origin ( $\gamma = 0$ ) then the image is a straight line which is parallel to the reflection in the real axis of the tangent to the original circle at the origin; if the original circle is a line that does not pass through the origin, its image is a circle.



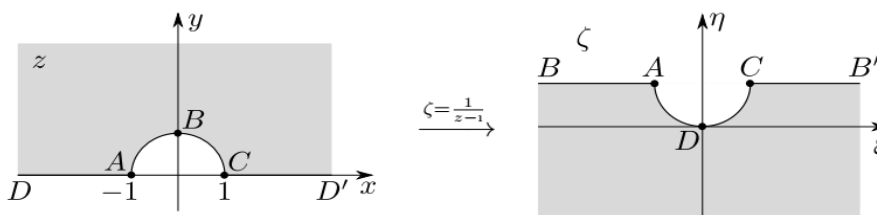
Note that the Möbius map is conformal at all points except the location of its pole, and takes the complex plane one-to-one into the complex plane.

It is not usually necessary to do detailed calculations when using Möbius maps on a domain bounded by straight lines and circles; it is enough to know that the boundary maps to a combination of straight lines and circles, and to know the angles where they meet: these latter are preserved by the map.

**Example.** The domain  $D$  consists of the upper half-plane  $y > 0$  with its intersection\* with the closed unit disc removed (thus, it has a semicircular indentation centred on the origin). Find the image of the domain under the map  $\zeta = 1/(z - i)$ , indicating the images of significant boundary points.

**Solution.** The image boundary is made up of straight lines and circles. The points  $z = \pm 1$  map to  $\zeta = 1/(\pm 1 - i) = (\pm 1 + i)/2$ , respectively,  $z = \infty$  maps to  $\zeta = 0$ , and  $z = i$  maps to  $\zeta = \infty$ . Therefore, the unit circle is mapped to the line  $\eta = 1/2$  and the  $x$ -axis is mapped to the circle  $|\zeta - i/2| = 1/2$ .

The image of the upper half-plane with the closed unit disc removed is then the lower half-plane  $\eta < 1/2$  with the closed disc  $|\zeta - i/2| \leq 1$  removed.

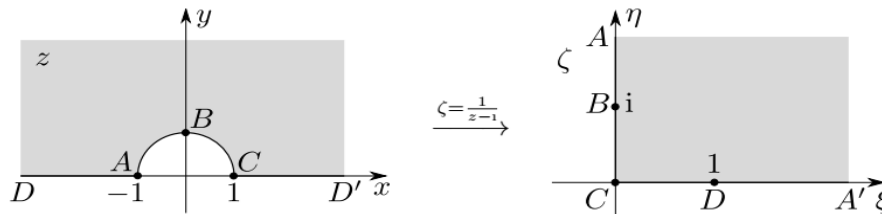


**Example.** Find a map from the domain of the previous example to the quarter plane  $\xi > 0, \eta > 0$ .

**Solution.** Start by looking at the angles. We have two right angles which will be preserved by a Möbius transform. We want one at  $\zeta = 0$  and one at  $\zeta = \infty$ . So set

$$\zeta = \frac{z - 1}{z + 1}.$$

Then  $z = 1$  is mapped to  $\zeta = 0$ ,  $z = \infty$  is mapped to  $\zeta = 1$ ,  $z = -1$  is mapped to  $\zeta = \infty$ , and  $z = i$  is mapped to  $\zeta = i$ .



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**Example.** Map the upper half-plane  $y > 0$  to the unit disc  $|\zeta| < 1$ .

**Solution.** The UHP is described by the inequality

$$\left| \frac{z - i}{z + i} \right| < 1$$

(the set of points closer to  $i$  than to  $-i$ ). This is exactly what we need: the map is

$$\zeta = \frac{z - i}{z + i}.$$

**Example.** If  $|w| < 1$ , show that the map

$$\zeta = \frac{z - w}{1 - \bar{w}z}$$

maps the unit disc one-to-one to itself.

**Solution.** The map is a Möbius map and is automatically one-to-one. We have

$$\begin{aligned} |z - w|^2 - |1 - \bar{w}z|^2 &= z\bar{z} - z\bar{w} - \bar{z}w + w\bar{w} - 1 + \bar{w}z + w\bar{z} - z\bar{z}w\bar{w} \\ &= -(1 - |z|^2)(1 - |w|^2); \end{aligned}$$

The last quantity is non-positive, and so

$$\left| \frac{z - w}{1 - \bar{w}z} \right| \leq 1,$$

with equality if and only if  $|z| = 1$ , so the unit circle maps to the unit circle. (Remember that  $|w| < 1$ ; what happens if  $|w| = 1$ ?)

It can be shown that, up to a rotation, the Möbius maps of the example above are the *only* one-to-one maps of the unit disc to itself. The two real parameters needed to define  $w$ , and the single real angle of the rotation, are the three free parameters in the Riemann mapping theorem, for having mapped a domain onto the unit disc, we can apply one of these Möbius maps to fix the images of three boundary points (or of an interior point and a direction at it).

**Example.** Map the upper half plane to itself, permuting the points  $0, 1$  and  $\infty$ . (Because the map preserves the orientation of the interior of the domain vis a vis its boundary, only the even permutations, in which  $(0, 1, \infty)$  are mapped to  $(1, \infty, 0)$  or  $(\infty, 0, 1)$  are possible.)

**Solution.** We need a Möbius map with real coefficients, so that the real axis maps to itself. First take  $(0, 1, \infty)$  to  $(1, \infty, 0)$ . This means that the map has its pole at  $z = 1$ , so the denominator is  $z - 1$  (or a multiple thereof); infinity maps to zero, so the numerator must be a constant (so the map looks like constant/ $z$  at infinity; finally, the remaining constant is fixed by the requirement that  $0$  maps to  $1$ , and we have

$$\zeta = \frac{1}{1 - z}.$$

The map taking  $(0, 1, \infty)$  to  $(\infty, 0, 1)$  is likewise found to be

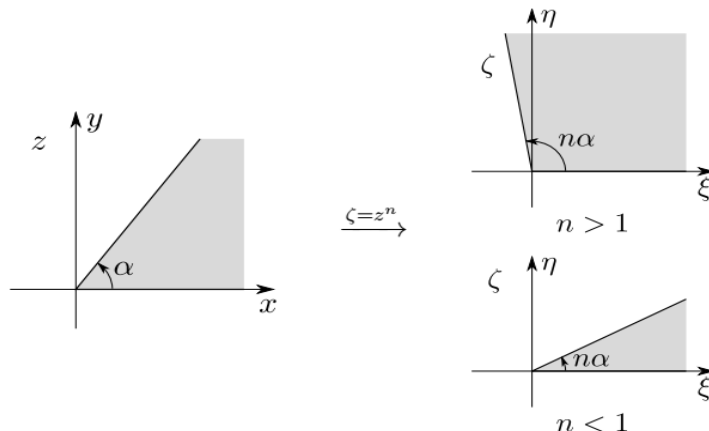
$$\zeta = \frac{z - 1}{z};$$

it has its pole at  $z = 0$ , to send that point to infinity, it vanishes at  $z = 1$ , sending that point to  $0$ , and is asymptotic to  $1$  at infinity as also required.

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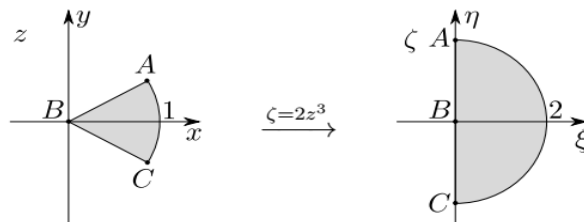
## Powers of $z$

The powers of  $z$  are of most interest when the boundary of  $D$  contains a corner at the origin. Suppose  $D$  is a sector  $0 < \arg z < \alpha$ ; then the image of this wedge under the map  $\zeta = z^n$  is the sector  $0 < \arg \zeta < n\alpha$ , provided of course that  $n\alpha < 2\pi$  so that the map is conformal. This multiplying of angles by  $n$  is often useful in getting rid of corners in  $\partial D$ , as the next two examples show. Note that there is no need for  $n$  to be an integer; if we want to multiply an angle by  $3/2$  we choose  $n = \frac{3}{2}$ , and if we want to halve an angle we choose  $n = \frac{1}{2}$ .



**Example.** The domain  $D$  consists of the sector  $-\frac{\pi}{6} < \arg z < \frac{\pi}{6}$  of the unit disc (a slice of pizza). Map it to a semicircle of radius 2.

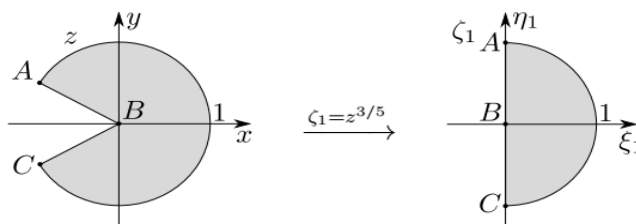
**Solution.** We need to get rid of the angle of  $\frac{\pi}{3}$  at the origin. The map  $\zeta_1 = z^3$  does exactly this; however it takes the unit circle to the unit circle. Hence the required map is  $\zeta = 2\zeta_1 = 2z^3$ .



**Example.** The domain  $D$  consists of the unit disc with the sector  $\frac{5\pi}{6} \leq \arg z \leq \frac{7\pi}{6}$  removed, so it looks like a partly eaten pizza (note the  $\leq$  which ensures that  $D$  is open). Find a map from  $D$  to the upper half-plane.

**Solution.** The boundary of  $D$  has three corners, with angles  $5\pi/3$  at the origin, and  $\frac{\pi}{2}$  on the unit circle. We need to get rid of all of these. We do the mapping in 3 stages.

1. First put  $\zeta_1 = z^{\frac{3}{5}}$ . This gets us to a semicircle  $-\pi/2 < \arg \zeta_1 < \pi/2$ ,  $0 < |\zeta_1| < 1$ .

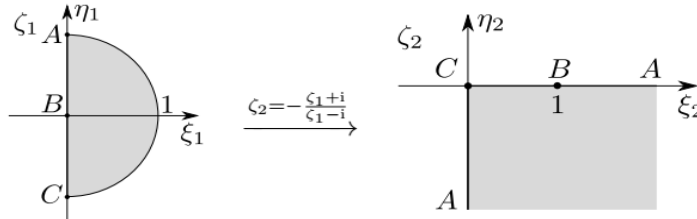


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2. Map the semi-circle in the  $\zeta_1$ -plane to a quarter plane in the  $\zeta_2$ -plane using a Möbius transformation. We have two right angles at  $\zeta_1 = \pm i$ . We choose the mapping such that  $\zeta_1 = -i$  is mapped to  $\zeta_2 = 0$  and  $\zeta_1 = i$  is mapped to  $\zeta_2 = \infty$ , so  $\zeta_2$  is a multiple of  $(\zeta_1 + i)/(\zeta_1 - i)$ . If we also specify that the image of  $\zeta_1 = 0$  is  $\zeta_2 = 1$  we find

$$\zeta_2 = -\frac{\zeta_1 + i}{\zeta_1 - i}$$

which maps the semi-circle to the first quadrant in the  $\zeta_2$ -plane.



3. The map  $\zeta = \zeta_2^2$  will map the quarter plane  $0 < \arg \zeta_2 < \pi/2$  to the upper half-plane  $\eta > 0$ . The combined map is

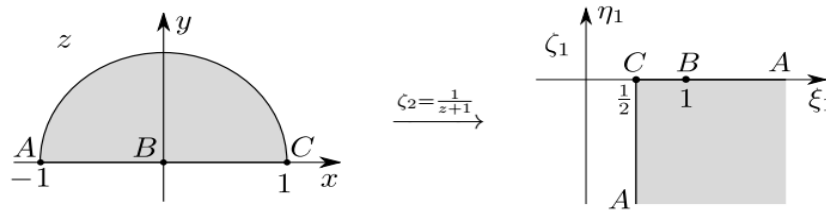
$$\zeta = \zeta_2^2 = \left(-\frac{\zeta_1 + i}{\zeta_1 - i}\right)^2 = \left(\frac{z^{3/5} + i}{z^{3/5} - i}\right)^2.$$

**Example.** Map  $D$ , the semicircle  $0 < |z| < 1$ ,  $0 < \arg z < \pi$ , to the upper half-plane. (This map is often useful; the example is similar to the previous one.)

**Solution.** First we get rid of the semicircle by setting

$$\zeta_1 = \xi_1 + i\eta_1 = \frac{1}{z+1}.$$

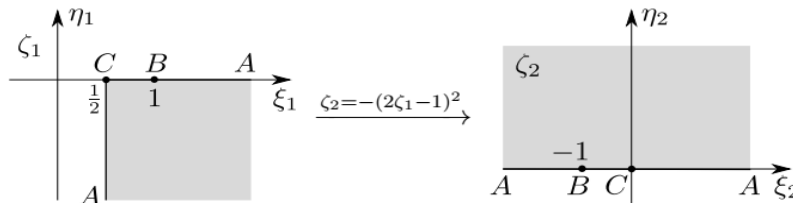
This inversion with respect to the left-hand corner point of  $D$  takes  $D$  to  $D_1$ , the quarter plane  $\xi_1 > 1/2$ ,  $\eta_1 < 0$ .



The quarter plane is mapped to the upper half plane by the transformation

$$\zeta = \zeta_2 = -(2\zeta_1 - 1)^2 = -\left(\frac{z-1}{z+1}\right)^2.$$

Note the minus sign, which gets us to the upper half plane.



Note also that we could have made this a bit slicker by starting with the map

$$\zeta_1 = \frac{z-1}{z+1},$$

which sends the corners of the semicircle to  $0$  and  $\infty$ , making the subsequent squaring easier.

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## Exponential and logarithm

Now consider the exponential function and its inverse, the logarithm. As  $e^{z+2\pi i} = e^z$ , the image of any horizontal strip of width  $2\pi$  is repeated infinitely often, once for each of the strips obtained by shifting the original one vertically by  $2\pi$ . Furthermore, as

$$\zeta = e^z = e^x e^{iy},$$

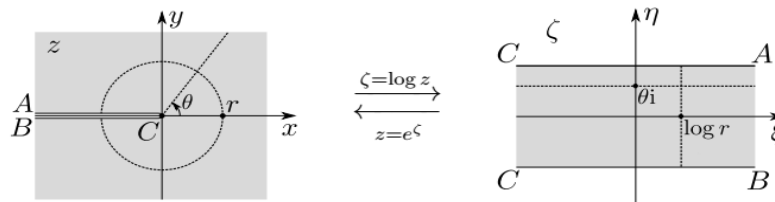
the image of the ‘baseline’ strip  $-\pi < y < \pi$ ,  $-\infty < x < \infty$  is the whole plane with the negative real ( $\xi$ ) axis removed.<sup>4</sup> The horizontal lines  $y = \text{constant}$  map onto rays  $\arg \zeta = y$ ,  $0 < |\zeta| < \infty$ , and the vertical line segments  $x = \text{constant}$ ,  $-\pi < y < \pi$  map to circles of radius  $e^x$  (with the point  $\zeta = -e^x$  missing). In particular, the imaginary axis is mapped to the unit circle.

Note also that the exponential map generates plane polar coordinates from a rectangular Cartesian grid.

The inverse map, the logarithm, is defined on the whole plane minus a cut from infinity to the origin (the cut is of course enforced by the multivalued nature of the exponential map). It takes this cut plane to a strip parallel to the  $\xi$  axis and of width  $2\pi$ ; if the cut is along  $\arg z = \alpha$ , the strip is  $\alpha - 2\pi < \eta < \alpha$ ,  $-\infty < \xi < \infty$ . since

$$\zeta = \xi + i\eta = \log z = \log |z| + i \arg z$$

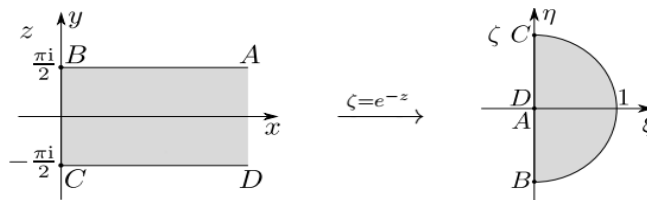
(with a suitable branch cut), circles  $|z| = \text{constant}$  map to lines  $\xi = \text{constant}$ , and rays  $\arg z = \text{constant}$  map to lines  $\eta = \text{constant}$ .



The exponential map is useful to open out a strip or half strip, and the log does the reverse. Note that neither exp nor log has a vanishing derivative.

**Example.** Map the half-strip  $-\frac{\pi}{2} < y < \frac{\pi}{2}$ ,  $0 < x < \infty$  onto the interior of a semicircle, with the point at infinity mapping to the origin and the short side of the strip to the semicircle.

**Solution.** If we try to use the exponential function directly, we shall send the point at infinity to infinity. Instead, put  $\zeta_1 = -z$ ,  $\zeta = e^{\zeta_1} = e^{-z}$ .

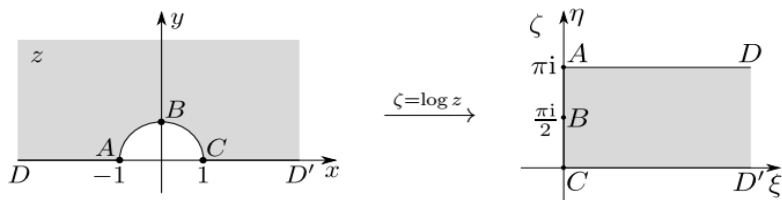


**Example.** The domain  $D$  consists of the upper half-plane with the upper half of the closed unit disc removed (see an earlier example in section ). Map it to a half-strip.

<sup>4</sup>If we include the image of the line  $y = -\pi$  as well, the image of the strip is  $\mathbb{C} \setminus \{0\}$ ; thus the image of the whole plane is the whole plane covered infinitely often: this is an example of Picard’s theorem in action. The function  $e^z$  has an essential singularity at infinity, and its lacunary value is 0.

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**Solution.** The logarithm springs into action. The boundary of  $D$  consists of segments of rays (mapping to segments of horizontal lines) and a segment of a circle centred at the origin (mapping to a vertical line). The rays map to horizontal lines and the semicircle to a segment of the imaginary axis. The image of  $D$  under  $\zeta = \log z$  is thus the half-strip  $0 < \xi < \infty, 0 < \eta < \pi$ .



### Sin, cos, tan

Unlike the exponential, the sine and cosine functions have derivatives that vanish; they are, however also periodic (and have essential singularities at infinity).<sup>5</sup> Hence these functions can combine the properties of opening out (doubling) angles on the boundary of  $D$  with the useful strip-mapping properties of the exponential. (Their inverses have similar uses to the logarithm.)

**Example.** Investigate the effect of the mapping  $\zeta = \cos z$  on the half-strip  $0 < y < \infty, 0 < x < \pi$ . Calculate the image of the lines  $y = \text{constant}$ .

**Solution.** The critical points are at  $z = 0, \pi$ . Hence the internal angles of  $\partial D$  are doubled there. Note that

$$\cos z = \cos x \cosh y + i \sin x \sinh y.$$

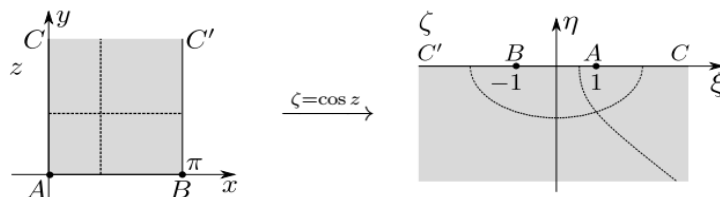
When  $z$  is real,  $\cos z$  runs from 1 (at  $z = 0$ ) to  $-1$  (at  $z = \pi$ ); thus this line maps to the interval  $-1 < \xi < 1$  of the  $\xi$  axis, and the image of  $D$  is below this line. The line  $x = 0, y > 0$  maps to  $\zeta = \cos iy = \cosh y$ , namely the  $\xi$  axis from 1 to  $+\infty$ , and the line  $x = \pi, y > 0$  maps to the segment  $-\infty < \xi = \cos(\pi + iy) = -\cosh y < -1$  of the  $\xi$  axis. Thus the map is from the half-strip to the lower half-plane, with the two critical points mapping to  $\zeta = \pm 1$ . As for a line  $y = \text{constant}$ , it maps to the curve given parametrically by  $\xi = \cos x \cosh y, \eta = \sin x \sinh y$  as  $x$  varies from  $-\pi$  to  $\pi$ , namely the ellipse

$$\frac{\xi^2}{\cosh^2 y} + \frac{\eta^2}{\sinh^2 y} = 1.$$

Similarly, the lines  $x = \text{constant}$  map to the hyperbolae

$$\frac{\xi^2}{\cos^2 x} - \frac{\eta^2}{\sin^2 x} = 1,$$

which are orthogonal to the ellipses; the coordinate system generated is the elliptic coordinate system.



**Example: the tangent function.** Show that the mapping  $\zeta = \tan z$  maps the strip  $-\frac{\pi}{4} < x < \frac{\pi}{4}, -\infty < y < \infty$  to the unit disc.

<sup>5</sup>They are periodic in one direction only, parallel to the real axis. It is an easy consequence of Liouville's theorem that if a function  $f(z)$  is holomorphic and periodic in two different directions (ie there are numbers  $a$  and  $b$ , corresponding to linearly independent vectors in the plane, such that  $f(z) = f(z+a) = f(z+b)$ ) then  $f(z)$  is constant. The next best thing is to allow the function to have a pole in each period parallelogram (formed by  $a$  and  $b$ ), and this leads to the fascinating theory of Jacobi elliptic functions.

## Lecture 3: Revision of conformal mapping

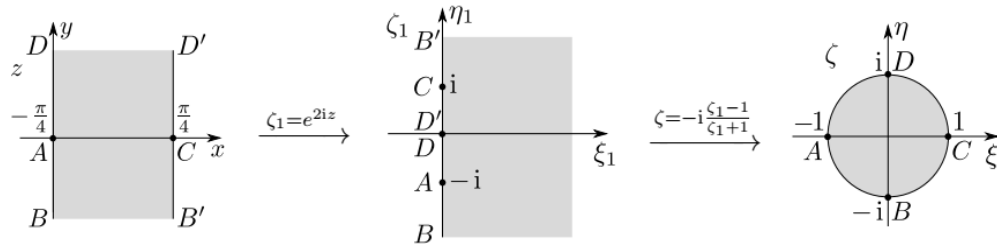
**Solution.** It would be a mistake to think of  $\tan z$  as the ratio  $\sin z/\cos z$ : the product of two conformal mappings has no natural geometric meaning. The composition does, however, and writing

$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{i} \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = -i \frac{e^{2iz} - 1}{e^{2iz} + 1}$$

shows that the sequence

$$\zeta_1 = e^{2iz}, \quad \zeta_2 = \frac{\zeta_1 - 1}{\zeta_1 + 1}, \quad \zeta = -i\zeta_2$$

can be used to analyse the map as illustrated.

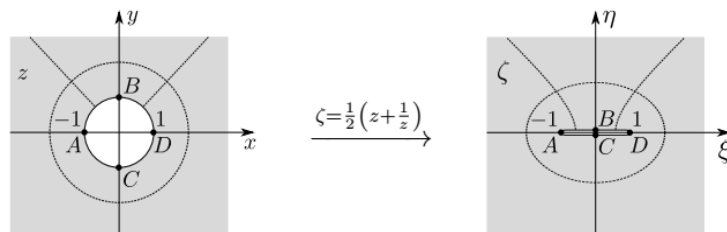


### The Joukowski map

Our final example is the map called the Joukowski map,

$$\zeta = \frac{1}{2} \left( z + \frac{1}{z} \right).$$

This has critical points at  $z = \pm 1$ . The image of the unit circle is the slit  $-1 < \xi < 1$  of the real axis, as when  $z = e^{i\theta}$ ,  $\zeta = \cos \theta$ . The exterior of the unit circle is mapped to the whole plane exterior to this slit (which is a branch cut for the inverse mapping); similarly the interior of the unit circle is mapped to the whole plane exterior to the same slit. A circle  $|z| = \rho > 1$  is mapped to an ellipse (exercise) and a ray  $\arg z = \text{constant}$  to a member of the orthogonal family, in this case hyperbolae.



The map is useful in simple models for inviscid flow (see below); in particular, the region exterior to a circle passing through one of the critical points is mapped to the exterior of an aerofoil-like shape.

