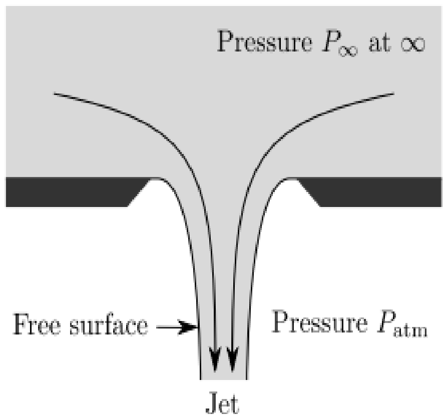
Lecture 6: Free streamline flow

We now move on to discuss potential flow with free surfaces. With only fixed (and known) boundaries we can find the potential whenever we can find the relevant conformal map. However, if there are *free* surfaces, these are unknown and we have to find them as part of the solution. We can still do this for a more restricted class of flows in which the fixed boundaries are straight lines (or circular arcs) when gravity and surface tension are neglected.

Example: Flow out of a slot



We need an extra condition on any free surface, in addition to the fact that in steady flow it is a streamline. This condition follows from Bernoulli's equation

$$\frac{p}{\rho} + \frac{1}{2}|\mathbf{u}|^2 = \text{constant}$$

in steady flow. On a free surface $p = p_{\text{atm}} = \text{constant}$, so

$$|\mathbf{u}|^2 = \text{constant}$$

in addition to the streamline condition

$$\psi = \text{constant}.$$

The goal is to specify the flow region in both the complex potential (w) plane and the complex velocity or hodograph ($w' = u - \mathrm{i}v$) plane. Recall that both are holomorphic functions. If we can find a map F from one to the other we have a relation

$$u - \mathrm{i}v = \frac{\mathrm{d}w}{\mathrm{d}z} = F(w),$$

from which

$$\int \frac{\mathrm{d}w}{F(w)} = \int \mathrm{d}z$$

gives the flow (implicitly).

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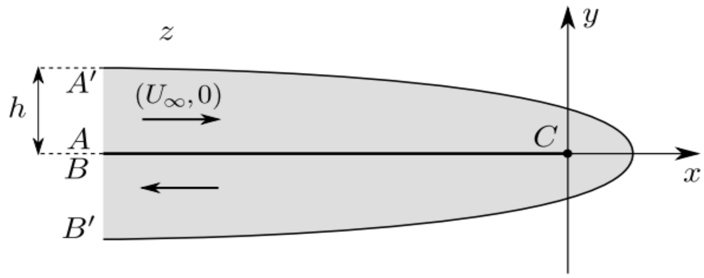
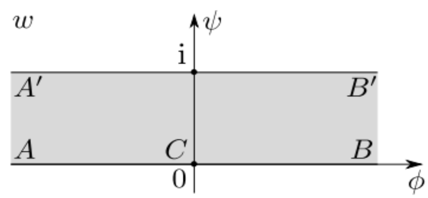
Example: Teapot flow

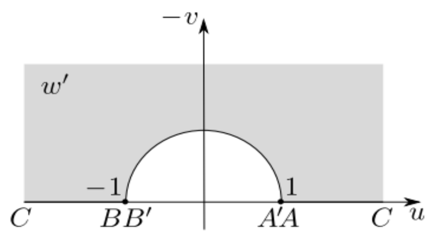


Far to the left above the plate $\mathbf{u} \sim (U_\infty, 0)$ and the layer depth is h. Note that the velocity is infinite at C ($w = \mathrm{O}(z^{1/2})$ as $z \to 0$). AB and $A'B'$ are both streamlines: set $\psi = 0$ on AB, then $\psi = U_\infty h$ on $A'B'$ because the flux of fluid through AA' is equal to $[\psi]_A^{A'}$. Scale $w(z)$ with $U_\infty h$ and z with h; then

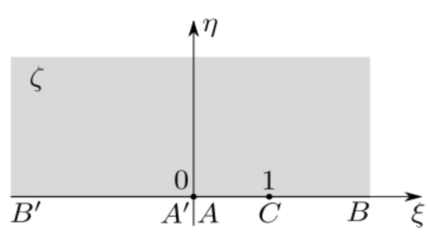
$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 = 1 \quad \text{on the free surface.}$$

Step 1 Construct potential and hodograph planes (take $w = 0$ at C w.l.o.g.).





Step 2 Map both to an auxillary ζ plane. Set $\zeta = e^{\pi w}$ ($A \to 0$, $B \to \infty$, $C \to 1$) and $\zeta = \big((w'-1)(w'+1)\big)^2$ ($A \to 0$, $B \to \infty$, $C \to 1$).



Hence

$$e^{\pi w} = \zeta = \left(\frac{w'-1}{w'+1}\right)^2,$$

giving

$$\frac{w'-1}{w'+1} = e^{\pi w/2}.$$

[Check whether this is the correct branch: at the "nose" $w = w' = \mathrm{i}$, giving

$$\frac{\mathrm{i}-1}{\mathrm{i}+1} = e^{\mathrm{i}\pi/2} = \mathrm{i}.]$$

Rearranging

$$w' = \frac{1 + e^{\pi w/2}}{1 - e^{\pi w/2}} = -\coth \pi w/4.$$

Integrating

$$\int \tanh \pi w/4 \, \mathrm{d}w = -z + \text{constant} \qquad \Rightarrow \qquad \frac{4}{\pi} \log \cosh \pi w/4 = -z$$

since $w = 0$ when $z = 0$. Thus

$$\cosh \pi w/4 = e^{-\pi z/4}.$$

The free streamline is given by $\psi = 1$,

$$e^{-\pi x/4} (\cos \pi y/4 - i \sin \pi y/4) = \cosh(\pi\phi/4 + \mathrm{i}\pi/4) = \frac{1}{\sqrt{2}} \cosh \pi\phi/4 + \frac{\mathrm{i}}{\sqrt{2}} \sinh \pi\phi/4.$$

Thus, parametrically,

$$e^{-\pi x/4} \cos \pi y/4 = \frac{1}{\sqrt{2}} \cosh \pi\phi/4, \qquad e^{-\pi x/4} \sin \pi y/4 = -\frac{1}{\sqrt{2}} \sinh \pi\phi/4.$$

Or, eliminating ϕ,

$$\frac{1}{2} = e^{-\pi x/2} \left(\cos^2 \pi y/4 - \sin^2 \pi y/4\right) = e^{-\pi x/2} \cos \pi y/2.$$

At the nose $y = 0$, obtain $x = \dfrac{2 \log 2}{\pi}$ — the "thinning factor".