Lecture 12: Complex Fourier Transforms

Recall that if f is periodic with period $2L$ then under certain regularity assumptions, *e.g.* f piecewise smooth,[6] the **Convergence Theorem for Fourier Series** states that

$$\frac{1}{2}\big(f(x_-)+f(x_+)\big)=\sum_{n=-\infty}^{\infty}c_n e^{-\mathrm{i}n\pi x/L}, \tag{20}$$

where $f(x_-)=\lim_{x\uparrow 0}f(x)$, $f(x_+)=\lim_{x\downarrow 0}f(x)$ and

$$c_n=\frac{1}{2L}\int_{-L}^{L}f(x)e^{\mathrm{i}n\pi x/L}\,\mathrm{d}x.$$

We can remove the requirement that f be periodic by letting $L\to\infty$. Writing $k=n\pi/L$ and $\lim_{L\to\infty}2Lc_n=\bar{f}(k)$ we find

$$\bar{f}(k)=\int_{-\infty}^{\infty}f(x)e^{\mathrm{i}kx}\,\mathrm{d}x$$

which is the **Fourier transform** of f. Remembering that

$$h\sum_n f(nh)\to\int f(k)\,\mathrm{d}k \qquad \text{as } h\to 0 \quad \text{(Riemann integral)}$$

we set $h=\pi/L$ to give in (20)

$$\frac{1}{2}\big(f(x_-)+f(x_+)\big)=\lim_{L\to\infty}\sum_{n=-\infty}^{\infty}\frac{\bar{f}(n\pi/L)}{2L}e^{-\mathrm{i}n\pi x/L}=\frac{1}{2\pi}\lim_{h\to 0}h\sum_{n=-\infty}^{\infty}\bar{f}(nh)e^{-\mathrm{i}nhx},$$

which may be shown, given sufficient regularity conditions on f (*e.g.* f piecewise smooth and $\int_{-\infty}^{\infty}|f(x)|\mathrm{d}x<\infty$), to lead to the **Fourier Inversion Theorem**, which states that

$$\frac{1}{2}\big(f(x_-)+f(x_+)\big)=\frac{1}{2\pi}\fint_{-\infty}^{\infty}\bar{f}(k)e^{-\mathrm{i}kx}\,\mathrm{d}k,$$

where the prinicipal value integral

$$\fint_{-\infty}^{\infty}=\lim_{R\to\infty}\int_{-R}^{R}.$$

Our aim is to generalize the Fourier Inversion Theorem to a wider class of functions by allowing $k\in\mathbb{C}$. In particular, we assume that f is continuous (for brevity) and $f=\mathrm{O}(\mathrm{e}^{c|x|})$ as $|x|\to\infty$ for some constant $c>0$ (this rules out $f=e^{x^2}$, for example).

To investigate the convergence of the Fourier integral at $x=\pm\infty$, we split up f by writing

$$f(x)=f_+(x)+f_-(x)$$

where

$$f_+(x)=0 \text{ for } x<0, \qquad f_-(x)=0 \text{ for } x>0.$$

Since

$$|\bar{f}_+(k)|=\left|\int_0^{\infty}f_+(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x\right|\leq\int_0^{\infty}|f(x)\mathrm{e}^{\mathrm{i}kx}|\mathrm{d}x\leq\int_0^{\infty}|f(x)|\mathrm{e}^{-Im(k)\,x}\mathrm{d}x<\infty$$

for $Im(k)>c$, $\bar{f}_+(k)$ exists and is holomorphic for $Im(k)>c$ (with $\bar{f}'_+(k)$ being the Fourier transform of $-\mathrm{i}xf_+(x)$).

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[6] f is piecewise smooth on $\mathbb{R}$ iff f and f' are piecewise continuous on all closed bounded subintervals of $\mathbb{R}$.

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Let $F_+(x) = \mathrm{e}^{-\alpha x} f_+(x)$, where $\alpha > c$, so that $\bar{F}_+(k) = \bar{f}_+(k + \mathrm{i}\alpha)$ exists and is holomorphic for $Im(k) > c - \alpha$. Since $\alpha > c$, the Fourier Inversion Theorem gives

$$F_+(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{F}_+(k) \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}k,$$

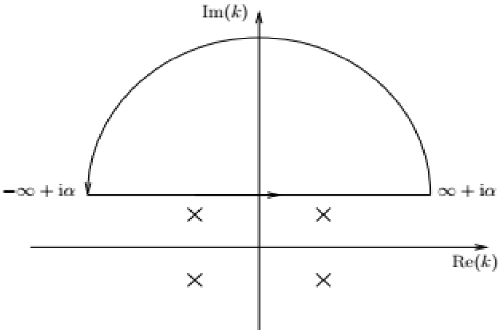
i.e.

$$\mathrm{e}^{-\alpha x} f_+(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}_+(k + i\alpha) \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}k$$

i.e.

$$f_+(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}_+(k + i\alpha) \mathrm{e}^{-\mathrm{i}(k+i\alpha)x} \mathrm{d}k = \frac{1}{2\pi} \int_{-\infty+\mathrm{i}\alpha}^{\infty+\mathrm{i}\alpha} \bar{f}_+(k) \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}k. \tag{21}$$

Suppose $\bar{f}_+(k)$ can be continued below $Im(k) = c$, so that it is holomorphic in some region $\Omega_+ \supset \{Im(k) > c\}$ except for singularities at $k = a_j$. By the deformation theorem, the inversion contour $\Gamma_+ = \{x + i\alpha : -\infty < x < \infty\}$ may be deformed into $\Omega_+ \backslash \{a_j\}$ provided it passes above the singularities a_j of $\bar{f}_+$. Since the singularities of $\bar{f}_+$ are below the inversion contour, for $x < 0$ we can deform the inversion contour to $+\mathrm{i}\infty$ to get zero, which we know is the value of $f_+(x)$ for $x < 0$. If there were singularities of $\bar{f}_+$ above the inversion contour then we would pick up contributions from these when we close the integral at $\mathrm{i}\infty$.



The same procedure works for $f_-(x)$ with everything upside down: $\bar{f}_-(k)$ exists and is holomorphic for $Im(k) < -c$, while an application of the Fourier Inversion Theorem to $F_-(x) = \mathrm{e}^{\beta x} f_-(x)$ gives

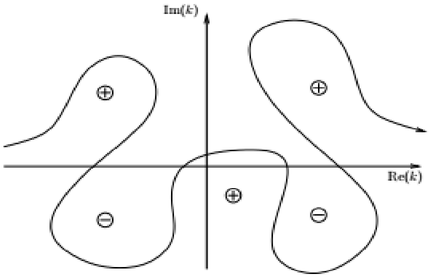
$$f_-(x) = \frac{1}{2\pi} \int_{-\infty-\mathrm{i}\beta}^{\infty-\mathrm{i}\beta} \bar{f}_-(k) \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}k$$

provided $-\beta < -c$. Suppose $\bar{f}_-(k)$ can be continued above $Im(k) = -c$, so that it is holomorphic in some region $\Omega_- \supset \{Im(k) < -c\}$ except for singularities at $k = b_j$. By the deformation theorem, the inversion contour $\Gamma_- = \{x - i\beta : -\infty < x < \infty\}$ may be deformed into $\Omega_- \backslash \{b_j\}$ provided it passes underneath the singularities b_j of $\bar{f}_-$.

If there is a non-empty overlap region $\Omega = \Omega_+ \cap \Omega_- \backslash (\{a_j\} \cup \{b_j\})$, then the Fourier Transform of f is defined by $\bar{f}(k) = \bar{f}_+(k) + \bar{f}_-(k)$ for $k \in \Omega$. Moreover, if Γ_+ and Γ_- can be deformed into the same contour $\Gamma \subset \Omega$, with Γ above (below) the the singularities of $\bar{f}_+$ ($\bar{f}_-$), as illustrated below, then

$$f(x) = \frac{1}{2\pi} \int_{\Gamma} \bar{f}(k) \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}k$$

by the deformation theorem; note that we need Γ to extend from $\Re(k) = -\infty$ to $\Re(k) = \infty$ and $\{a_j\} \cap \{b_j\}$ to be empty.



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Example: Fourier transform of the Heaviside function

The Heaviside function

$$H(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x > 0, \end{cases}$$

has Fourier transform

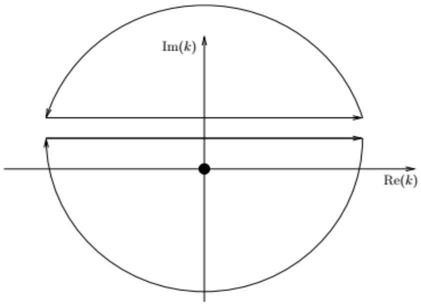
$$\bar{H}(k) = \int_0^\infty e^{ikx}\, dx = \left[\frac{e^{ikx}}{ik}\right]_0^\infty = -\frac{1}{ik}$$

provided $Im(k) > 0$. We can continue $\bar{H}(k)$ into $\mathbb{C}/\{0\}$ because $-1/ik$ is holomorphic except for a simple pole at $k = 0$. Since $H = 1_+$, when we invert the inversion contour must have $Im(k) > 0$:

$$H(x) = -\frac{1}{2\pi i}\int_{-\infty+i\alpha}^{\infty+i\alpha} \frac{e^{-ikx}}{k}\, dk \qquad (\alpha > 0).$$

Now, if $x < 0$ we can close the contour in the upper half plane to find by Cauchy's Theorem that $H(x) = 0$ for $x < 0$. For $x > 0$ we need to close the contour in the lower half plane, and we pick up a residue contribution from the pole at the origin (note the minus sign since we are integrating clockwise round the pole) to find

$$H(x) = -2\pi i \times \left(-\frac{1}{2\pi i}\right) = 1 \qquad \text{for } x > 0.$$



Laplace transforms

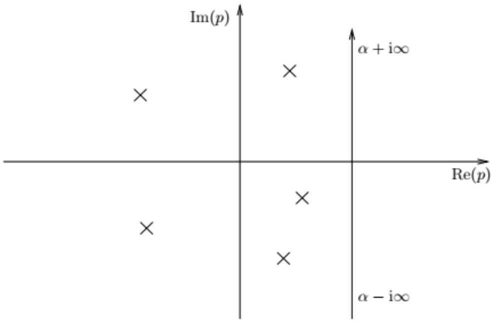
When we set $k = ip$ (p complex) and $\bar{f}_+(k) = \hat{f}_+(p)$, we get

$$f_+(x) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \hat{f}_+(p)e^{px}\, dp$$

where

$$\hat{f}_+(p) = \int_0^\infty f_+(x)e^{-px}\, dx,$$

which is just the Laplace transform and inversion formula. Now α must be sufficiently large that the inversion contour lies to the right of any singularities of $\hat{f}(p)$.



So Laplace transforms are just a special case of Fourier Transforms if you allow complex k.