Chapter 2

Vector Algebra

2.1. Vectors. Basic Definitions

– Vector (definition)
– Collinear and Coplanar Vectors
– Equality of Vectors
–Linear operations (addition, scalar multiplication)
– Basis in Vector Space
– Projection of a Vector onto Axis
– Coordinates of a Vector

Vectors and Scalars

Suppose, you asked somebody the question how old he is? The answer “18” will completely satisfy you. You’ll find this answer exhaustive. But if you ask how to get to the Polytechnic Institute and hear in response that you need to walk 700 meters, you will have to ask another question: “In which direction?”

Some of the things we measure are determined simply by their magnitudes. Quantities that can be defined only by their numerical value are called “scalar quantities” or “scalars”. For example, time, length of the line, mass, work, charge, energy, temperature, etc.

Not all quantities are like this. We need more information to describe a force, displacement, or velocity of the moving object. To describe, for example, a force, we need to record the direction in which it acts as well as how large it is. In physics quantities which have both *a magnitude* and *a direction* in space, are called “*vector quantities*” or “*vectors*”.

The term *vector* derives from the Latin word *vehere,* meaning “to carry.” The idea is that if you were to carry something from one point to another the trip could be represented by the directed line segment.

A vector in geometry is *a directed line segment* with a tail A, called *the initial point*, and a tip B, called *the terminal point* (Fig. 2.1).

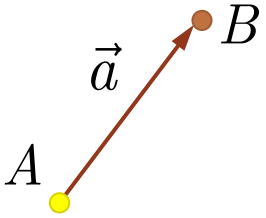

Fig. 2.1

An abstract *mathematical vector* $\vec{a}$ we’ll consider as *an ordered pair of points* A and B.

Visually, a vector we will indicate by an arrow.

A directed line segment $\overrightarrow{AB}$ that represents a vector $\vec{a}$ we will write in the form

$$\overrightarrow{AB} = \vec{a}.$$

The length (*the norm*) of a vector $\overrightarrow{AB} = \vec{a}$ is defined as the distance between points A and B and is denoted as $\left|\vec{a}\right|, \left|\overrightarrow{AB}\right|$.

If the initial point and the terminal point of a vector coincide, this vector is called the *Zero Vector* and is denoted as $\vec{0}$. The zero vector is special because it is the only vector with the length of zero. The zero vector is also unique because it is the only vector that does not have a direction.

A vector of *the length one* is called *a Unit Vector*. The unit vectors often serve as markers for various directions (unit vectors may also be called “versors”).

Definition 2.1.

The vectors lying on the same line or on the parallel lines are called the collinear vectors (Fig.2.2).

The collinear vectors $\vec{a}$ and $\vec{b}$ we will denote as $\vec{a} \parallel \vec{b}$.

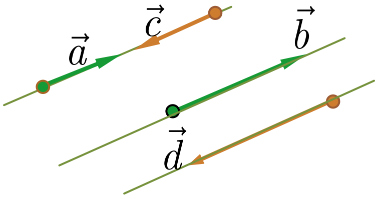

Fig. 2.2

Collinear nonzero vectors can have the same or opposite directions.

On the Fig. 2.2 the vectors $\vec{a}$ and $\vec{b}$; $\vec{c}$ and $\vec{d}$ have the same direction (we denote as $\vec{a} \uparrow\uparrow \vec{b}$, $\vec{c} \uparrow\uparrow \vec{d}$), the vectors $\vec{a}$ and $\vec{c}$, $\vec{b}$ and $\vec{d}$ have the opposite directions (we denote as $\vec{a} \uparrow\downarrow \vec{c}$, $\vec{b} \uparrow\downarrow \vec{d}$).

Definition 2.2.

The vectors lying in the plane (or are parallel to the same plane) are called the coplanar vectors (Fig. 2.3).

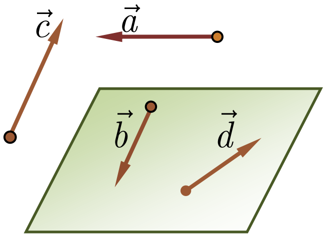

Fig. 2.3

Definition 2.3.

Two vectors $\vec{a}$ and $\vec{b}$ *are considered to be* *equal* *if they have the same length and the same direction*:

$$\vec{a}=\vec{b} \Leftrightarrow \begin{cases} \vec{a} \parallel \vec{b}, \\ \vec{a} \uparrow\uparrow \vec{b}, \\ \left|\vec{a}\right| = \left|\vec{b}\right|. \end{cases}$$

Geometrically vectors are equal if they are parallel transformation of one another, thus vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ (Fig. 2.4) are equal even though they are in different positions.

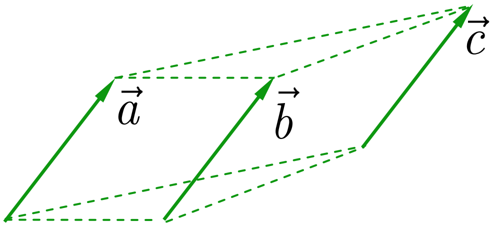

Fig. 2.4

Such vectors are called *free* vectors. You can move a free vector anywhere you'd like as long as you don't change its length or direction, and it remains the same vector. For example, the vector of the velocity of the elevator car (Fig. 2.5) can be transferred to any point [6].

However, such transformations are permissible not always, for instance, the velocity $\vec{v}$ of particles of water of a mountain waterfall at any moment (Fig. 2.6). It is impossible to say that the velocity of the waterfall at any other point is the same. In many physics and engineering problems, you'll be dealing with vectors that apply *at a given location*; such vectors are called *linked* (*bound*, *attached*, *ancored*) vectors. You're not allowed to relocate bound vectors as you can free vectors.

If we consider the velocity of wire of any cross section to uniformly raise the load (Fig. 2.7), now it is possible to transfer this vector along the line of action of the force of tension. Such vectors are called *sliding* vectors, they are useful for problems involving torque and angular motion.

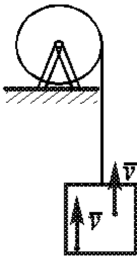

Fig. 2.5

Fig. 2.6

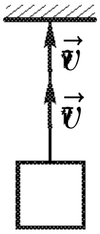

Fig. 2.7

Vector Algebra Operations

Two principal operations involving vectors are *scalar multiplication* and *vector addition*.

Vectors can be multiplied by real numbers (*the scalars*). Let us consider $\vec{a} \neq \vec{0}$ and $\lambda > 0$. Then $\lambda\vec{a}$ is a vector whose length is λ times the length of $\vec{a}$ and this vector preserves the direction of $\vec{a}$ (Fig. 2.8). In the same manner, $(-\lambda\vec{a})$ is a vector pointing in the direction opposite to that of $\vec{a}$ but λ times as long as $\vec{a}$.

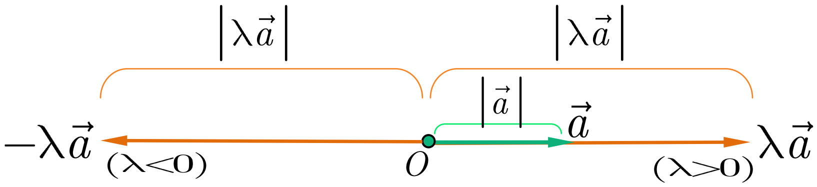

Fig. 2.8

We call this operation *scalar multiplication* (*vector scaling*) (Fig. 2.9).

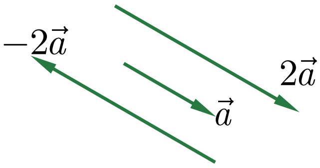

Fig. 2.9

The scalar multiplication satisfies two properties.

The first says that if $\vec{a}$ is any vector and λ and μ are real numbers, then

$$\lambda \cdot (\mu \cdot \vec{a}) = (\lambda\mu) \cdot \vec{a}\,.$$

The second property is totally obvious from the Fig. 2.8:

$$1 \cdot \vec{a} = \vec{a}\,.$$

Observe that if the vector $\vec{a}$ and the scalar λ are nonzero then the vectors $\vec{a}$ and $\lambda\vec{a}$ lie on the same line if their initial points coincide, and lie on parallel or coincident lines if they do not.

N.B. *Thus we say that vectors* $\vec{a}$ *and* $\lambda\vec{a}$ *are collinear vectors.*

Claim 2.1. *The vectors* $\vec{a}$ *and* $\vec{b}$ *are collinear if and only if there exists such scalar* λ *that* $\vec{b} = \lambda\vec{a}$*:*

$$\vec{b} \parallel \vec{a} \Leftrightarrow \vec{b} = \lambda\vec{a}\,.$$

Let us consider the vector

$$\vec{a}^{\,0} = \frac{1}{\left|\vec{a}\right|}\vec{a}, \quad \vec{a} \neq \vec{0},\ \lambda = \frac{1}{\left|\vec{a}\right|} > 0\,.$$

The length of the vector $\vec{a}^{\,0}$ is equal to one ($\left|\vec{a}^{\,0}\right| = 1$), vectors $\vec{a}$ and $\vec{a}^{\,0}$ have the same direction ($\vec{a} \uparrow\uparrow \vec{a}^{\,0}$). The vector $\vec{a}^{\,0}$ is *a unit vector* of $\vec{a}$ (*in the direction of the nonzero vector* $\vec{a}$) (Fig. 2.10).

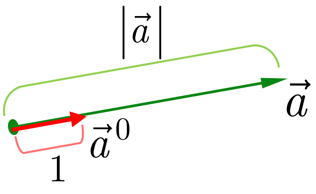

Fig. 2.10

And we may write:

N.B.

$$\vec{a} = \left|\vec{a}\right| \cdot \vec{a}^{\,0}.$$

The vector $(-\vec{a})$ is called *an opposite* (*negative*) to the vector $\vec{a}$, this vector has the length of the vector $\vec{a}$ (Fig. 2.11).

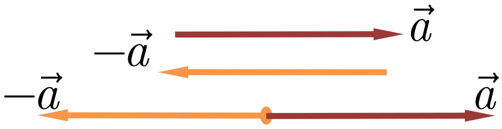

Fig. 2.11

You would be familiar from the study of physics (about forces) with the fact that *vectors can be added.*

Indeed, if $\vec{a}$ and $\vec{b}$ are vectors in the plane then their sum is the diagonal from the common initial point O to the opposite vertex in the parallelogram defined by $\vec{a}$ and $\vec{b}$ as in Fig. 2.12 (*the Parallelogram Law of Addition*). This law has a simple physical application: if two puppies drag a bone in different directions with different forces $\vec{a}$ and $\vec{b}$, then their total effort will be directed along the diagonal of the parallelogram.

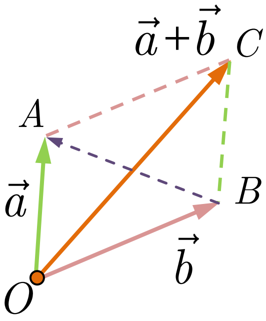

Fig. 2.12

This operation is called *vector addition*. Physicists call vector $\vec{a}+\vec{b}$ as the *resultant vector* of the vectors $\vec{a}$ and $\vec{b}$.

We can also add vectors $\vec{a}$ and $\vec{b}$ by translating vector $\vec{b}$ such that the tail of $\vec{b}$ coincides with the tip of $\vec{a}$; the vector from the tail of $\vec{a}$ to the tip of the translated $\vec{b}$ then represents the sum of the vectors $\vec{a}$ and $\vec{b}$ (Fig. 2.13). This procedure is called *the Triangle Law* or *the Nose-to-Tail Law*.

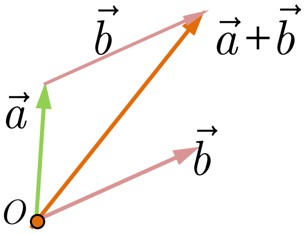

Fig. 2.13

When more than two vectors are being added the “nose-to-tail” method is used.

The sum of the finite number of vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ *is the vector initial point of which coincides with the initial point of the first vector* $\vec{a}_1$ *and its terminal point coincides with the terminal point of the last vector* $\vec{a}_n$ (Fig. 2.14).

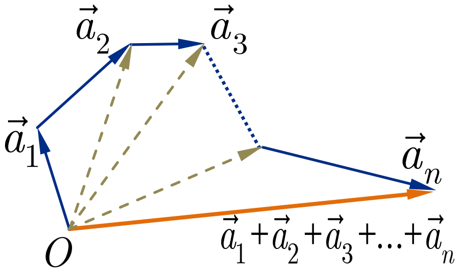

Fig.2.14

It follows from the Fig. 2.12 that

$$\vec{a}+\vec{b}=\vec{b}+\vec{a},$$

so that we get that *vector addition* is *commutative*.

Vector addition is also *associative*. This means that, as can be seen in the Fig. 2.15, when adding three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ it does not matter whether we first add $\vec{a}$ and $\vec{b}$ and add $\vec{c}$ to the result: $(\vec{a}+\vec{b})+\vec{c}$ or whether we first add $\vec{b}$ and $\vec{c}$ and add the result to $\vec{a}$: $\vec{a}+(\vec{b}+\vec{c})$.

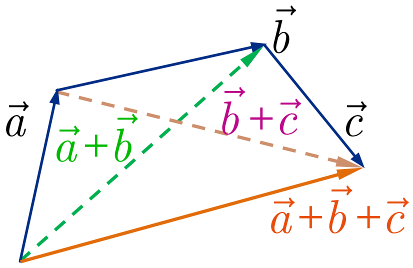

Fig. 2.15

Another easy property of vector addition is the existence of a vector $\vec{0}$ such that when added to any vector $\vec{a}$ gives back $\vec{a}$ again; that is, for all vectors $\vec{a}$

$$\vec{a} + \vec{0} = \vec{a}\,.$$

Similarly, given any vector $\vec{a}$ there is an opposite vector $(-\vec{a})$ which obeys

$$\vec{a} + (-\vec{a}) = \vec{0}\,.$$

We will often employ *the difference* $\vec{a} - \vec{b}$ between vectors $\vec{a}$ and $\vec{b}$ to denote $\vec{a} + (-\vec{b})$ (Fig. 2.16).

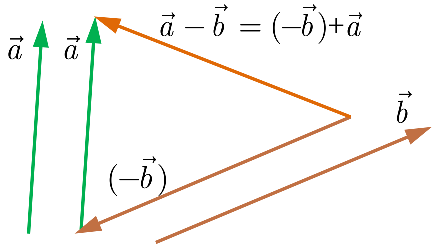

Fig. 2.16

In the parallelogram of Fig. 2.12, notice that diagonal AC, as we know, is the sum $\vec{a} + \vec{b}$ of the vectors $\vec{a}$ and $\vec{b}$; the other diagonal BA (directed as shown) is the difference $\vec{a} - \vec{b}$ of vectors $\vec{a}$ and $\vec{b}$.

We define that $\lambda\vec{a} = 0$ if $\vec{a} = \vec{0}$ or $\lambda = 0$.

Finally, notice that scalar multiplication and addition are compatible: scalar multiplication and addition can be performed in any order:

$$\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b}$$

and

$$(\lambda + \mu)\cdot\vec{a} = \lambda\cdot\vec{a} + \mu\cdot\vec{a}\,.$$

The former identity says that *scalar multiplication is distributive over vector addition.*

Vectors live in vector spaces. Formally, a vector space is just a set of vectors, together with the operations of addition and scalar multiplication, satisfying the above properties.

Linear Dependence and Linear Independence of Vectors

Let us consider a set of vectors

$$\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$$

and let's form by taking arbitrary constants $\alpha_1, \alpha_2, \dots, \alpha_n$ the sum

$$\vec{x} = \alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 + \dots + \alpha_n \vec{e}_n ,$$

which is called *the linear combination of vectors*.

One can then say that the vector $\vec{x}$ is the linear combination of vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ (or $\vec{x}$ is linearly decomposed in vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$).

Definition 2.4.

A set of vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ is called a linearly dependent, if there exist constants $\alpha_1, \alpha_2, \dots, \alpha_n$, not all zero, such that

$$\alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 + \dots + \alpha_n \vec{e}_n = \vec{0}$$

$(\alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2 \neq 0)$.

The vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ form a linearly independent set (briefly, they are called linearly independent), if

$$\alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 + \dots + \alpha_n \vec{e}_n = \vec{0}$$

implies that

$$\alpha_1 = \alpha_2 = \dots = \alpha_n = 0 .$$

For example, let us consider collinear vectors $\vec{a}$ and $\vec{b}$:

$$\vec{a} \parallel \vec{b} \Rightarrow \vec{a} = \lambda \vec{b} \Rightarrow \vec{a} - \lambda \vec{b} = \vec{0},$$

so

$$\alpha_1 = 1, \ \alpha_2 = -\lambda ,$$

and **N.B.** *collinear vectors are linearly dependent.*

Theorem 2.1.

The vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ *are linearly dependent if and only if one of the vectors is the linear combination of the preceding vectors.*

Consider the main properties of linearly dependent and linearly independent vectors.

Property 1

If the system of vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ *contains zero vector then the system is linearly dependent.*

Property 2

If the subsystem $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_k$ *of the system of vectors* $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ *is the linearly dependent system, then the whole system of vectors is linearly dependent too.*

Property 3

If we add a finite number of vectors to the linearly dependent system of vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$, *then the new system is linearly dependent too.*

Property 4

If the system of vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ *is linearly independent, then any part of this system is independent too.*

Property 5

1. *One vector is linearly dependent (independent) if and only if it is zero (non-zero):*

$$\alpha \cdot \vec{0} = \vec{0},\ \forall\alpha \ \ (\forall\vec{e} \neq \vec{0} : \alpha \cdot \vec{e} = \vec{0} \Leftrightarrow \alpha = 0).$$

2. *The system of two vectors is linearly dependent (independent) if and only if the vectors are collinear (non-collinear).*

3. *The system of three vectors is linearly dependent (independent) if and only if they are coplanar (non-coplanar).*

Let us introduce such a concept as a basis of a linear space.

Definition 2.5.

A basis of a linear space is called any linear independent system with the maximum possible amount of vectors. The number of vectors of the basis is called the dimension of the space.

Thus, a basis of a linear space is a system of linearly independent vectors, every vector of a linear space can be expressed as a linear combination of members of the basis.

The linear space is called *a finite dimensional* (denoted as $\mathbb{R}^n$), if its basis contains the finite number of vectors (n).

N.B. *A linear space can have different bases, but any basis of the linear space $\mathbb{R}^n$ consists of n elements.*

Let us give some examples.

1. *Any nonzero vector forms a basis on the line* (in the space $\mathbb{R}^1$) (Fig. 2.17).

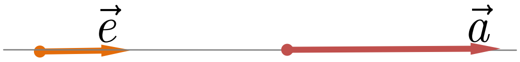

Fig. 2.17

Vector $\vec{e}$ forms a basis on the line because $\vec{e} \neq \vec{0}$, and any another vector $\vec{a}$ on this line is collinear to $\vec{e}$ $(\vec{a} \parallel \vec{e})$, that's why

$$\vec{a} = x\vec{e}\,,\ x \in \mathbb{R}\,.$$

It means that a vector $\vec{a}$ is expressed by the vector $\vec{e}$.

2. *Any two non-collinear vectors form the basis on the plane* (*in two- dimensional space* $\mathbb{R}^2$).

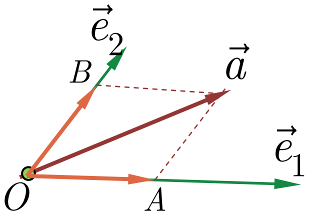

Fig. 2.18

Two vectors $\vec{e}_1$ and $\vec{e}_2$ (Fig. 2.18) are non-collinear, so they are linearly independent.

Let's consider any vector $\vec{a}$ on the plane (see Fig. 2.18).

We have

$$\vec{a} = \overrightarrow{OA} + \overrightarrow{OB}\,,$$

$$\overrightarrow{OA} \parallel \vec{e}_1 \Rightarrow \overrightarrow{OA} = x\vec{e}_1,\ \ \overrightarrow{OB} \parallel \vec{e}_2 \Rightarrow \overrightarrow{OB} = y\vec{e}_2, x, y \in \mathbb{R}\,,$$

and

$$\vec{a} = x\vec{e}_1 + y\vec{e}_2\,,$$

i.e. vector $\vec{a}$ is a linear combination of vectors $\vec{e}_1$ and $\vec{e}_2$.

Thus, vectors $\vec{e}_1$ and $\vec{e}_2$ form basis in two-dimensional space $\mathbb{R}^2$.

3. *Any three non-coplanar vectors form the basis in the three-dimensional space* $\mathbb{R}^3$.

For any vector $\vec{a} \neq \vec{0}$, and for any three non-coplanar vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ in space $\mathbb{R}^3$ we have

$$\vec{a} = x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3, \; x, y, z \in \mathbb{R}.$$

The coefficients in a linear combination are called the coordinates of a vector in a basis.

Claim 2.2.

1. *In the given basis equal vectors have the same coordinates.*
2. *The coordinates of the sum of vectors in the given basis are equal to the sum of the coordinates of these vectors.*
3. *When we multiply a vector by a scalar in the given basis we multiply each coordinate of the vector by this scalar.*

► 1. Suppose that

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n,$$

$$\vec{y} = y_1\vec{e}_1 + y_2\vec{e}_2 + \ldots + y_n\vec{e}_n.$$

The equality of vectors ($\vec{x} = \vec{y}$) implies that

$$x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n = y_1\vec{e}_1 + y_2\vec{e}_2 + \ldots + y_n\vec{e}_n,$$

or

$$(x_1 - y_1)\vec{e}_1 + (x_2 - y_2)\vec{e}_2 + \ldots + (x_n - y_n)\vec{e}_n = \vec{0}.$$

Since the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ are linearly independent we have

$$x_1 - y_1 = x_2 - y_2 = \ldots = x_n - y_n = 0 \Rightarrow$$

$$x_1 = y_1, \, x_2 = y_2, \ldots, x_n = y_n.$$

2. $\vec{x} + \vec{y} = (x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n) + (y_1\vec{e}_1 + y_2\vec{e}_2 + \ldots + y_n\vec{e}_n) =$

$$= (x_1 + y_1)\vec{e}_1 + (x_2 + y_2)\vec{e}_2 + \ldots + (x_n + y_n)\vec{e}_n.$$

3. $\lambda\vec{x} = \lambda(x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n) = (\lambda x_1)\vec{e}_1 + (\lambda x_2)\vec{e}_2 + \ldots + (\lambda x_n)\vec{e}_n$. ◄

Projection of a Vector onto Axis

The word "projection" comes from the Latin "projectio" — “throwing forward”. The idea of this concept arose, apparently, when observing shadows that cast illuminated objects (Fig. 2.19).

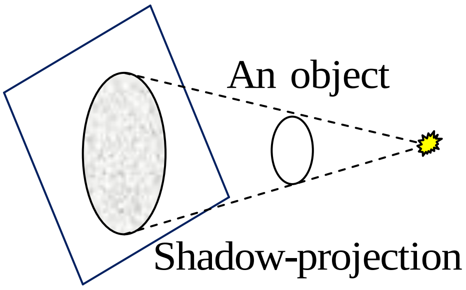

Fig. 2.19

Let us consider at first the most simple, but nonetheless important for understanding, a special case of projection.

The orthogonal projection of the point M of the space or plane on the line L we call the point M' of the intersection of the straight line and the plane (Fig. 2.20 a)) or the line (Fig. 2.20 b)), which passes through the point M, perpendicular to the line L.

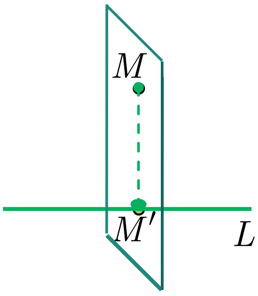

Fig. 2.20 a)

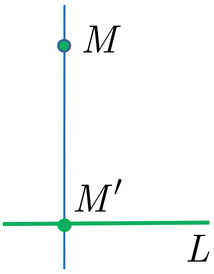

Fig. 2.20 b)

Let the given axis u with the direction *unit vector* $\vec{e}$.

Let us consider the vector $\vec{a} = \overrightarrow{AB}$. The orthogonal projections of the points A and B on the axis u are respectively the points A' and B' (Fig. 2.21).

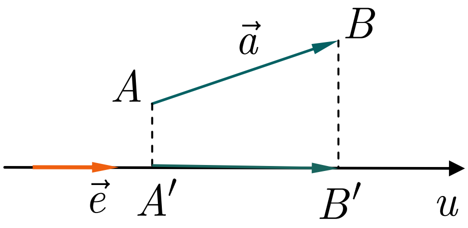

Fig. 2.21

The vector $\overrightarrow{A'B'}$ is called *the Vector projection of the vector* $\vec a = \overrightarrow{AB}$ onto an axis u. We'll denote the vector projection *of a vector* $\vec a = \overrightarrow{AB}$ *onto an axis* u *with direction unit vector* $\vec e$ as

$$\overrightarrow{pr}_u \vec a = \overrightarrow{pr}_u \overrightarrow{AB}.$$

Definition 2.6.

The Scalar Projection (*Projection*) *of a vector* $\vec a = \overrightarrow{AB}$ *onto an axis* u *with direction unit vector* $\vec e$ (*the projection of the vector* $\vec a$ *on the direction of the vector* $\vec e$) *is called a scalar*

$$\mathrm{pr}_u \vec a = \mathrm{pr}_{\vec e} \vec a = \begin{cases} +\left|\overrightarrow{A'B'}\right|, & \overrightarrow{A'B'} \uparrow\uparrow \vec e, \\ -\left|\overrightarrow{A'B'}\right|, & \overrightarrow{A'B'} \uparrow\downarrow \vec e. \end{cases}$$

Since

$$\overrightarrow{A'B'} \parallel \vec e \Rightarrow \overrightarrow{A'B'} = x\vec e \Rightarrow \quad |x| = \frac{\left|\overrightarrow{A'B'}\right|}{|\vec e|} = \left|\overrightarrow{A'B'}\right|, \ |\vec e| = 1,$$

and the coordinate x in the basis $\vec e$ equals:

$$x = \pm\left|\overrightarrow{A'B'}\right| = \mathrm{pr}_u \overrightarrow{AB}.$$

Corollary

N.B. *The coordinate* x *of a vector* $\vec a$ *in the basis* $\vec e$ *is a projection of this vector onto the axis with direction unit vector* $\vec e$.

The following claim is true.

Claim 2.3.

The projection of a vector $\vec{a}$ onto an axis u is equal to the product of the length of the vector $\vec{a}$ by cosine of the angle between vector $\vec{a}$ and the axis:

$$\mathrm{pr}_u\vec{a} = \left|\vec{a}\right|\cos(\vec{a},u) = \left|\vec{a}\right|\cos(\vec{a},\vec{e}). \tag{2.1}$$

Note that the angle between vector and axis is the angle between this vector and the direction vector of axis.
And the angle φ between two vectors satisfies $0 \le \varphi \le \pi$.

▶ Let us prove this statement. Consider the following cases.

1) $0 < \varphi < \dfrac{\pi}{2}$ (Fig. 2.22 a)):

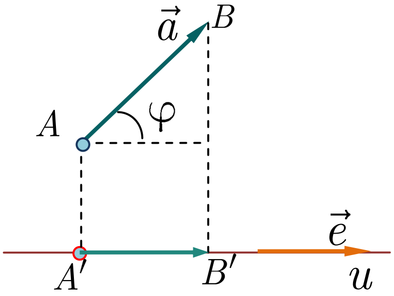

Fig. 2.22 a)

$$\mathrm{pr}_u\vec{a} = \left|\overrightarrow{A'B'}\right| = \left|\overrightarrow{AB}\right|\cos\varphi = \left|\vec{a}\right|\cos\varphi;$$

2) $\dfrac{\pi}{2} < \varphi < \pi$ (Fig. 2.22 b)):

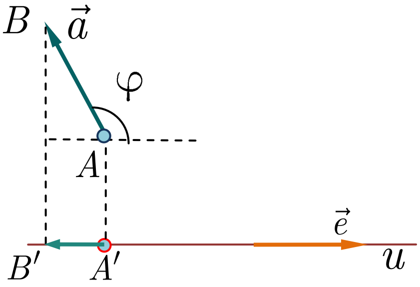

Fig. 2.22 b)

$$\mathrm{pr}_u\vec{a} = -\left|\overrightarrow{A'B'}\right| = -\left|\overrightarrow{AB}\right|\cos(\pi-\varphi) = \left|\overrightarrow{AB}\right|\cos\varphi = \left|\vec{a}\right|\cos\varphi;$$

for cases:

$$3)\ \varphi = 0; \qquad 4)\ \varphi = \frac{\pi}{2}; \qquad 5)\ \varphi = \pi$$

try to prove yourself.

The main properties of the projection of a vector on the axis are that linear operations on vectors lead to corresponding linear operations with the projections of these vectors.

Claim 2.4.

1. *The equal vectors have the equal projections on the same axis.*

2. *The projection of the sum of the vectors on an axis is equal to the sum of the projections of these vectors on the same axis:*

$$\text{pr}_u(\vec{a} + \vec{b}) = \text{pr}_u\vec{a} + \text{pr}_u\vec{b}\,.$$

3. *If we multiply a vector on a scalar, then the projection of a vector on an axis also be multiplied by this scalar:*

$$\text{pr}_u\lambda\vec{a} = \lambda\text{pr}_u\vec{a}\,.$$

► 1. Indeed, if $\vec{a} = \vec{b}$ then $|\vec{a}| = |\vec{b}|$ and $(\vec{a}, u) = (\vec{b}, u)$ and the formula (2.1) implies

$$\text{pr}_u\vec{a} = \text{pr}_u\vec{b}\,.$$

2. Let us construct OAB with $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$ (Fig. 2.23). Then we project the points O, A, B on the axis u, we will get points O', A', B'.

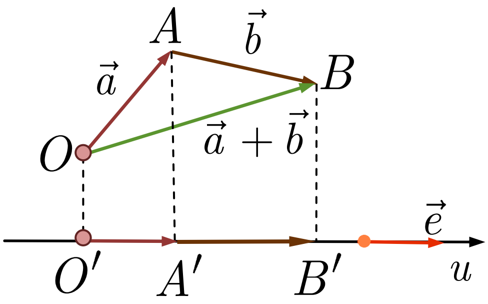

Fig. 2.23

Consider

$$x = \text{pr}_u \vec{a} \quad \text{and} \quad y = \text{pr}_u \vec{b}\,.$$

Then

$$\overrightarrow{O'A'} = x\vec{e}\,,\ \overrightarrow{A'B'} = y\vec{e}\,.$$

Hence,

$$\overrightarrow{O'B'} = (x+y)\vec{e}\,.$$

But this equality means that $x+y$ is the projection of the vector $\overrightarrow{OB} = \vec{a}+\vec{b}$, i.e.

$$\text{pr}_u(\vec{a}+\vec{b}) = \text{pr}_u\vec{a} + \text{pr}_u\vec{b}\,.$$

3. The equality is evident when $\lambda = 0$.

Let the angle between vector $\vec{a} = \overrightarrow{AB}$ and the axis u be α.

Suppose that $\lambda > 0$. Then the vector $\lambda\vec{a}$ forms with the axis L the same angle α.

Therefore, we get

$$\text{pr}_u\lambda\vec{a} = \left|\lambda\vec{a}\right|\cos\alpha = \left|\lambda\overrightarrow{AB}\right|\cos\alpha = \lambda\left|\overrightarrow{AB}\right|\cos\alpha = \lambda\text{pr}_u\vec{a}\,.$$

When $\lambda < 0$, then the vector $\lambda\vec{a}$ forms the angle $(\pi-\alpha)$ with the axis u. And we have

$$\text{pr}_u(\lambda\vec{a}) = \left|\lambda\overrightarrow{AB}\right|\cos(\pi-\alpha) = -\lambda\left|\overrightarrow{AB}\right|\cos(\pi-\alpha) = \lambda\left|\overrightarrow{AB}\right|\cos\alpha = \lambda\text{pr}_u\vec{a}\,. \blacktriangleleft$$

Vector Decomposition into Horizontal and Vertical Components

Consider two vectors $\vec{a}$ and $\vec{e}$, let the angle between them is φ (Fig. 2.24).

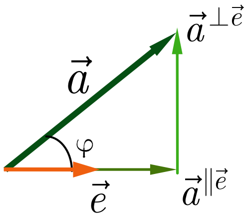

Fig. 2.24

A vector $\vec{a}$ can be resolved in the direction of the vector $\vec{e}$ into two components such that the vector addition of the components is equal to the original vector. The two components usually taken are a *horizontal component* $\vec{a}^{\|\vec{e}}$ and a *vertical component* $\vec{a}^{\perp\vec{e}}$. They are parallel and perpendicular to $\vec{e}$, respectively, such that

$$\vec{a} = \vec{a}^{\|\vec{e}} + \vec{a}^{\perp\vec{e}}.$$

We sometimes refer to parallel component $\vec{a}^{\|\vec{e}}$ as the result of projecting $\vec{a}$ onto $\vec{e}$ (*geometrical projection*), that's why

$$\vec{a}^{\|\vec{e}} = \overrightarrow{pr_{\vec{e}}\vec{a}} = \frac{\left|\vec{a}^{\|\vec{e}}\right|}{\left|\vec{e}\right|}\vec{e} = \left|\vec{a}^{\|\vec{e}}\right|\vec{e}^{\,0}.$$

Using elementary trigonometry, we have

$$\left|\vec{a}^{\|\vec{e}}\right| = \left|\vec{a}\right|\cos\varphi.$$

Therefore,

$$\vec{a}^{\|\vec{e}} = \frac{\left|\vec{a}\right|\cos\varphi}{\left|\vec{e}\right|}\vec{e}.$$

If $\vec{e}$ is a unit vector then we get

$$\vec{a}^{\|\vec{e}} = \left|\vec{a}^{\|\vec{e}}\right|\cdot\vec{e} = \left|\vec{a}\right|\cos\varphi\cdot\vec{e}.$$

Once we know $\vec{a}^{\|\vec{e}}$ we can easily solve for $\vec{a}^{\perp\vec{e}}$:

$$\vec{a}^{\perp\vec{e}} = \vec{a} - \vec{a}^{\|\vec{e}}, \quad \left|\vec{a}^{\perp\vec{e}}\right| = \left|\vec{a}\right|\sin\varphi.$$