We are continuing to study Vector Algebra.

2.2. Non-linear Operations with Vectors

- *Scalar (Dot) Product of Two Vectors (definition, basic properties, applications)*
- *Cross (Vector) Product of Two Vectors (definition, basic properties, applications)*
- *Triple Scalar Product of Three Vectors (definition, basic properties, applications)*

2.2.1. Scalar Product of Two Vectors

One of the ways in which two vectors can be combined is known as the scalar product. When we calculate the scalar product of two vectors the result, as the name suggests, is a scalar.

We will now learn how to calculate the scalar product and meet some geometrical applications.

Definition of Scalar Product

Let us study two vectors $\vec{a}$ and $\vec{b}$ drawn in Fig. 2.25. We draw these vectors so that their tails are at the same point. The angle between the two vectors has been labeled φ.

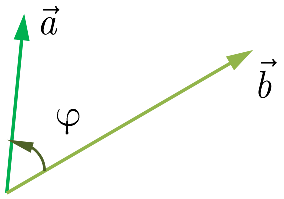

Fig. 2.25

Definition 2.7.

The scalar product of two vectors $\vec{a}$ and $\vec{b}$ is called the number, that is equal to the product of the length of these vectors by cosine of the angle between the vectors:

$$(\vec{a},\vec{b}) = |\vec{a}||\vec{b}|\cos\varphi, \quad \varphi = (\vec{a},\vec{b}).$$

Scalar product is also denoted as $\vec{a}\cdot\vec{b}$. So we some times refer to the scalar product as *the dot* (*inner*) *product*.

As we know,

$$|\vec{b}|\cos\varphi = \mathrm{pr}_{\vec{a}}\vec{b}\,, \qquad |\vec{a}|\cos\varphi = \mathrm{pr}_{\vec{b}}\vec{a}\,.$$

So the definition 2.7 of scalar product implies that

$$(\vec{a},\vec{b}) = |\vec{a}|\mathrm{pr}_{\vec{a}}\vec{b} = |\vec{b}|\mathrm{pr}_{\vec{b}}\vec{a}\,.$$

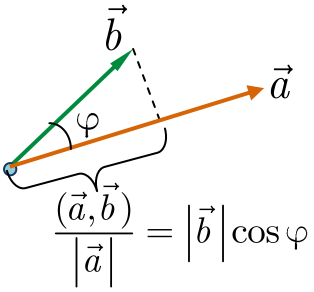

Fig. 2.26

We observe, since $|\cos\varphi| \le 1$, that

$$|(\vec{a},\vec{b})| \le |\vec{a}||\vec{b}|.$$

We get the basic inequality — *Cauchy–Schwarz inequality*.

We observe from the definition that the scalar product is a measure of how well $\vec{a}$ and $\vec{b}$ align with each other.

If $\vec{a}$ and $\vec{b}$ are nonzero, then the sign of the scalar product $\vec{a}\cdot\vec{b}$ is the same as the sign of the $\cos\varphi$: $\vec{a}\cdot\vec{b} > 0$ when φ is acute (Fig. 2.27 a)); $\vec{a}\cdot\vec{b} < 0$ when φ is obtuse (Fig. 2.27 b)).

a) b)

Fig. 2.27

If $\varphi = (\vec{a},\vec{b}) = \dfrac{\pi}{2}$, i.e. two vectors are *perpendicular* $(\vec{a} \perp \vec{b}\,)$ — maximally unaligned — since $\cos\dfrac{\pi}{2} = 0$ then $(\vec{a},\vec{b}) = 0$ (Fig. 2.28 a)).

The vectors $\vec{a}$ and $\vec{b}$ are said to be orthogonal when $(\vec{a},\vec{b}) = 0$.

From this definition, it follows that the zero vector $\vec{0}$ is orthogonal to every vector $\vec{a}$, because $(\vec{a},\vec{0}) = 0$.

If $\varphi = 0$ (see Fig. 2.28 b)) the two vectors are *collinear*, they have the same direction $(\vec{a} \uparrow\uparrow \vec{b}\,)$ — maximally aligned — and since $\cos 0 = 1$ then $(\vec{a},\vec{b}) = |\vec{a}||\vec{b}|$.

a) b)

Fig. 2.28

The Cauchy–Schwarz inequality implies that the value of the scalar product between two vectors is largest if the two vectors are collinear and have the same direction.

Basic Properties of Scalar Product

The scalar product has some important properties.

For any vectors $\vec{a}$, $\vec{b}$ and for arbitrary scalars α, β are valid the following equalities:

1) $$(\vec{a},\vec{b}) = (\vec{b},\vec{a})$$

(*symmetric of scalar product*);

2) $$(\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2,\vec{b}) = \alpha_1(\vec{a}_1,\vec{b}) + \alpha_2(\vec{a}_2,\vec{b}),$$

$$(\vec{a},\beta_1\vec{b}_1 + \beta_2\vec{b}_2) = \beta_1(\vec{a},\vec{b}_1) + \beta_2(\vec{a},\vec{b}_2)$$

(*bilinear* (*linear in both vectors*) *of scalar product*);

3) $$(\vec{a},\vec{a}) = \left|\vec{a}\right|^2 \geq 0, \qquad (\vec{a},\vec{a}) = \vec{0} \Leftrightarrow \vec{a} = \vec{0}$$

(*positive definite of scalar product*).

Example

Let's simplify the expression

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}).$$

❍ Using the bilinear property of scalar product, we get

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = (\vec{a},\vec{a}) + (\vec{b},\vec{a}) - (\vec{a},\vec{b}) - (\vec{b},\vec{b}).$$

As the scalar product is symmetric we obtain

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = (\vec{a},\vec{a}) - (\vec{b},\vec{b}).$$

By positive definite of scalar product we have

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = \left|\vec{a}\right|^2 - \left|\vec{b}\right|^2. \quad \bullet$$

Applications of Scalar Product

Now we will look at some ways in which the scalar product can be used.

1. *Calculating the length of a vector.*

From the properties of the scalar product we have:

$$\left|\vec{a}\right| = \sqrt{(\vec{a},\vec{a})}\,.$$

N.B. $$\sqrt{(\vec{a},\vec{a})} \neq \vec{a}\,.$$

2. *Finding the angle between two vectors.*

One of the most fundamental problems concerning vectors is that of computing the angle between two given vectors. It has numerous applications in mathematics and other sciences. In physics, it plays a role in the decomposition of forces into component forces that act in various directions. In computer science, it is useful for creating two-dimensional visualizations of three-dimensional objects.

From the definition of the scalar product we obtain an expression for $\cos(\vec{a},\vec{b})$:

$$\cos(\vec{a},\vec{b}) = \frac{(\vec{a},\vec{b})}{\left|\vec{a}\right|\left|\vec{b}\right|}.$$

Example

Let's find the angle φ between two vectors

$$\vec{p} = \vec{a} + \vec{b} \quad \text{and} \quad \vec{q} = \vec{a} - \vec{b}\,,$$

if

$$\left|\vec{a}\right| = \sqrt{3}\,,\ \left|\vec{b}\right| = 1,\ \varphi = (\vec{a},\vec{b}) = \frac{\pi}{6}.$$

❍ As we know

$$\cos(\vec{p},\vec{q}) = \frac{(\vec{p},\vec{q})}{\left|\vec{p}\right|\left|\vec{q}\right|}.$$

Let us find the scalar product and the length of given vectors $\vec{p}$ and $\vec{q}$:

$$\vec{p}\cdot\vec{q} = (\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = \left|\vec{a}\right|^2 - \left|\vec{b}\right|^2 = 3-1=2;$$

$$\left|\vec{p}\right| = \left|\vec{a}+\vec{b}\right| = \sqrt{(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})} = \sqrt{\vec{a}^2+2\vec{a}\cdot\vec{b}+\vec{b}^2} =$$

$$= \sqrt{\left|\vec{a}\right|^2 + 2\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cos\varphi + \left|\vec{b}\right|^2} = \sqrt{3+2\cdot\sqrt{3}\cdot 1\cdot\frac{\sqrt{3}}{2}+1} = \sqrt{7};$$

$$\left|\vec{q}\right| = \left|\vec{a}-\vec{b}\right| = \sqrt{(\vec{a}-\vec{b})(\vec{a}-\vec{b})} = \sqrt{\vec{a}^2-2\vec{a}\cdot\vec{b}+\vec{b}^2} = \sqrt{3-3+1} = 1.$$

Thus,

$$\cos(\vec{p},\vec{q}) = \frac{2}{\sqrt{7}\cdot 1} \Rightarrow (\vec{p},\vec{q}) = \arccos\frac{2}{\sqrt{7}}. \quad \bullet$$

3. *Testing orthogonality condition of two vectors.*

The scalar product of two non-zero vectors is equal to zero if and only if these vectors are orthogonal (the angle between them is $\frac{\pi}{2}$):

$$(\vec{a},\vec{b}) = 0 \Leftrightarrow (\widehat{\vec{a},\vec{b}}) = \frac{\pi}{2}, \qquad \vec{a},\vec{b} \neq \vec{0}.$$

4. *Finding the projection of vector $\vec{b}$ onto vector $\vec{a}$:*

Since

$$(\vec{a},\vec{b}) = \left|\vec{b}\right| \mathrm{pr}_{\vec{b}}\vec{a},$$

then

$$\mathrm{pr}_{\vec{b}}\vec{a} = \frac{(\vec{a},\vec{b})}{\left|\vec{b}\right|}.$$

Remark

As we know, vector $\vec{a}$ *can be resolved in the direction of vector* $\vec{b}$ *into two components: a horizontal component* $\vec{a}^{\parallel\vec{b}}$ *and a vertical component* $\vec{a}^{\perp\vec{b}}$. *In the previous lecture we obtained the formula for horizontal component*

$$\vec{a}^{\parallel\vec{b}} = \frac{\left|\vec{a}\right|\cos\varphi}{\left|\vec{b}\right|}\vec{b}\,.$$

It can be presented, using scalar product, in the alternative form

$$\vec{a}^{\parallel\vec{b}} = \frac{\left|\vec{a}\right|\left|\vec{b}\right|\cos\varphi}{\left|\vec{b}\right|^2}\vec{b} = \frac{(\vec{a},\vec{b})}{\left|\vec{b}\right|^2}\vec{b} = \frac{(\vec{a},\vec{b})}{(\vec{b},\vec{b})}\vec{b}\,.$$

5. *Finding the work done by the force.*

Let some particle is moving along the straight line from the point A to the point B under the constant force $\vec{F}$ that acts at an angle φ to the direction of motion (Fig. 2.29).

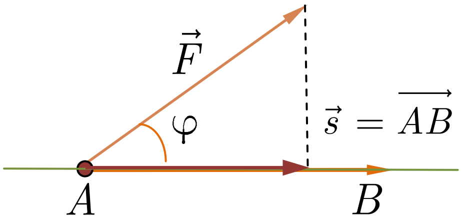

Fig. 2.29

The work A done by the force $\vec{F}$ for displacement $\vec{s}$ we find by the formula:

$$\mathrm{A} = \left|\vec{F}\right|\left|\vec{s}\right|\cos\varphi = (\vec{F},\vec{s}).$$

2.2.2. Cross (Vector) Product of Two Vectors

Geometrical Orientation in Geometrical Spaces

1. *Orientation on the line* [6].

On the horizontal line the positive direction is defined from left to right (Fig. 2.30) and on the vertical line the positive direction is bottom up (see Fig. 2.30).

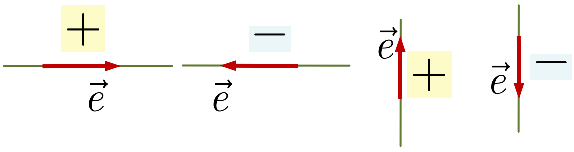

Fig. 2.30

On the line (space $\mathbb{R}^1$) we fix an arbitrary basis $\{\vec{e}\}$, $\vec{e} \neq \vec{0}$. All bases $\{\lambda\vec{e}\}$, $\lambda > 0$, we assume positive and all bases $\{\lambda\vec{e}\}$, $\lambda < 0$, we consider as negative.

2. *Orientation in 2-dimension.*

We will say that the parallelogram is built on the vectors $\bar{a} = \overline{OA}$ and $\bar{b} = \overline{OB}$, if the directed segments $\overline{OA}$ and $\overline{OB}$ form its adjacent sides. The parallelogram is called the oriented if it was built on the ordered two vectors $\vec{a}$ and $\vec{b}$. The orientation is assumed to be positive if «the shortest turn» from the first vector to the second vector occurs counterclockwise, in the opposite case it is negative direction.

In space $\mathbb{R}^2$ — on the plane — the basis $\{\vec{e}_1, \vec{e}_2\}$ represents the positive orientation if the shortest turn from $\vec{e}_1$ to $\vec{e}_2$ occurs counterclockwise, and represents the negative orientation if the turn occurs clockwise (Fig. 2.31).

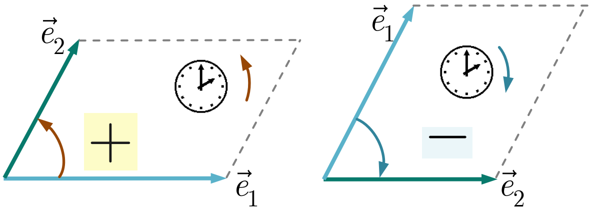

Fig. 2.31

3. *Orientation in 3-dimension.*

Point your thumb in the direction of $\vec{e}_1$, the first vector in the cross product. Now point your fingers in the direction of $\vec{e}_2$. Your palm will face in the direction of $\vec{e}_3$. In space $\mathbb{R}^3$ the basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ represents *the positive* (*right*) *orientation* if the shortest transfer from the vector $\vec{e}_1$ to the vector $\vec{e}_2$ occurs counterclockwise when you look at these vectors from the terminal point of the vector $\vec{e}_3$, and represents the *negative* (*left*) *orientation* if the shortest transfer from the vector $\vec{e}_1$ to the vector $\vec{e}_2$ occurs clockwise (Fig. 2.32). The vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ of the right basis form *the right triplet*, and the vectors of the left basis form *the left triplet*.

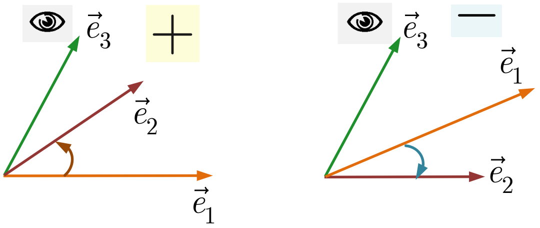

Fig. 2.32

Another way in which two vectors can be combined is known as the vector product. When we find the vector product of two vectors the result, as the name suggests, is a vector.

Definition of Vector (Cross) Product

Definition 2.8.

The vector (*cross*, *outer*) *product of two vectors* $\vec{a}$ *and* $\vec{b}$ *is called the vector* $\vec{c}$ *such that:*

1) *the length of the vector* $\bar{c}$ *is defined by the formula*

$$\left|\vec{c}\right| = \left|\vec{a}\right|\left|\vec{b}\right|\sin(\vec{a}, \vec{b}),$$

2) *the vector* $\vec{c}$ *is orthogonal to both vectors* $\vec{a}$ *and* $\vec{b}$, *i.e. it is orthogonal to the plane defined by* $\vec{a}$ *and* $\vec{b}$;

3) *the vector* $\vec{c}$ *is directed that the vectors* $\vec{a}$, $\vec{b}$ *and* $\vec{c}$ *form the right-handed triplet* (Fig. 2.33).

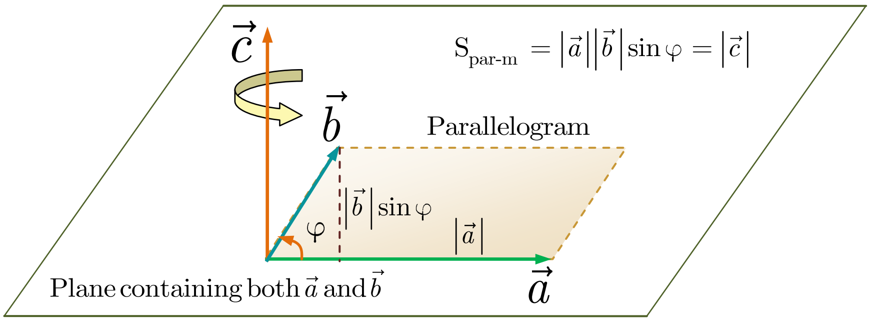

Fig. 2.33

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is denoted as

$$[\vec{a}, \vec{b}] \quad \text{or} \quad \vec{a} \times \vec{b}$$

(like the dot product, the term "cross product" comes from the symbol used in the last notation).

The cross product of the collinear vectors is equal to zero.

Basic Properties of Cross Product

Property 1

The length of the vector $[\vec{a}, \vec{b}]$ *is equal to the area of the parallelogram with adjacent sides* $\vec{a}$ *and* $\vec{b}$ (see Fig. 2.33):

$$\left|[\vec{a}, \vec{b}]\right| = S_{\text{par-m}}$$

Property 2

Criterion of the collinear of two vectors:
the vectors $\vec{a}$ *and* $\vec{b}$ *are collinear if and only if their cross product is equal to zero*:

$$\vec{a} \parallel \vec{b} \Leftrightarrow [\vec{a}, \vec{b}] = \vec{0}.$$

Property 3

The vector product is anti-commutative:

$$[\vec{a},\vec{b}] = -[\vec{b},\vec{a}]$$

The vector product is bilinear:

for any two vectors $\vec{a}$, $\vec{b}$ *and for arbitrary scalars are true the following equalities*

$$[\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2,\vec{b}] = \alpha_1[\vec{a}_1,\vec{b}] + \alpha_2[\vec{a}_2,\vec{b}],$$

$$[\vec{a},\beta_1\vec{b}_1 + \beta_2\vec{b}_2] = \beta_1[\vec{a},\vec{b}_1] + \beta_2[\vec{a},\vec{b}_2].$$

Property 4

Example

Let's simplify the expression

$$[(\vec{a}+\vec{b}),(\vec{a}-\vec{b})].$$

- Using bilinear property of vector product (property 4), property 2:

$$[\vec{a},\vec{a}] = [\vec{b},\vec{b}] = \vec{0}$$

and anti-commutative property 3:

$$[\vec{b},\vec{a}] = -[\vec{a},\vec{b}],$$

we obtain that

$$[(\vec{a}+\vec{b}),(\vec{a}-\vec{b})] = [\vec{a},\vec{a}] - [\vec{a},\vec{b}] + [\vec{b},\vec{a}] - [\vec{b},\vec{b}] = -2\cdot[\vec{a},\vec{b}]$$

An interesting conclusion can be drawn from obtained equality.

The absolute value of cross product $\left|[\vec{a},\vec{b}]\right|$ is equal, as stated above, to the area of the parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$.

Since vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$, as we know, are the diagonals of this parallelogram, consequently, the absolute value of cross product $\left|[(\vec{a}+\vec{b}),(\vec{a}-\vec{b})]\right|$ is equal to the area of the parallelogram determined by these diagonals.

Thus,

the square of the parallelogram determined by diagonals of the given parallelogram equals to the double square of this parallelogram.

$$\left|[(\vec{a}+\vec{b}),(\vec{a}-\vec{b})]\right| = 2\cdot\left|[\vec{a},\vec{b}]\right|. \quad \bullet$$

Example

Let's prove a fundamental relationship between the dot and cross products

$$\left|[\vec{a},\vec{b}]\right|^2 = \left|\vec{a}\right|^2\left|\vec{b}\right|^2 - (\vec{a},\vec{b})^2,$$

which is called the Lagrange Identity.

❍ As we know, if φ is the angle between $\vec{a}$ and $\vec{b}$, then

the length of the vector $[\vec{a},\vec{b}]$ is calculated by the formula

$$\left|[\vec{a},\vec{b}]\right| = \left|\vec{a}\right|\left|\vec{b}\right|\sin\varphi,$$

and the scalar product is calculated by the formula

$$(\vec{a},\vec{b}) = \left|\vec{a}\right|\left|\vec{b}\right|\cos\varphi.$$

Thus, we have

$$\left|[\vec{a},\vec{b}]\right|^2 = \left|\vec{a}\right|^2\left|\vec{b}\right|^2\sin^2\varphi = \left|\vec{a}\right|^2\left|\vec{b}\right|^2(1-\cos^2\varphi) =$$

$$= \left|\vec{a}\right|^2\left|\vec{b}\right|^2 - \left|\vec{a}\right|^2\left|\vec{b}\right|^2\cos^2\varphi = \left|\vec{a}\right|^2\left|\vec{b}\right|^2 - (\left|\vec{a}\right|\left|\vec{b}\right|\cos\varphi)^2 = \left|\vec{a}\right|^2\left|\vec{b}\right|^2 - (\vec{a},\vec{b})^2,$$

i.e.

$$\left|[\vec{a},\vec{b}]\right|^2 = \left|\vec{a}\right|^2\left|\vec{b}\right|^2 - (\vec{a},\vec{b})^2$$

Q.E.D. ●

Applications of Cross Product of Vectors

We will look at some ways in which the cross product can be used.

1. *Calculating the area of a parallelogram.*

Property 1 of the vector product implies that the area of a parallelogram determined by vectors $\vec{a}$ and $\vec{b}$ is calculated by the formula:

$$S_{\square} = \left|[\vec{a}, \vec{b}]\right|.$$

Area of the triangle, which is built on the vectors $\vec{a}$ and $\vec{b}$, is calculated by the formula:

$$S_{\triangle} = \frac{1}{2}\left|[\vec{a}, \vec{b}]\right|.$$

Example

Let's calculate the area of triangle, constructed on the vectors

$$\vec{a} = \vec{m} + 3\vec{n} \textit{ and } \vec{b} = 2\vec{m} + \vec{n},$$

if

$$|\vec{m}| = 4,\ |\vec{n}| = 1,\ (\vec{m}, \vec{n}) = \frac{\pi}{6}.$$

❍ We have

$$S_{\triangle} = \frac{1}{2}\left|[\vec{a}, \vec{b}]\right| = \frac{1}{2}\left|[\vec{m} + 3\vec{n}, 2\vec{m} + \vec{n}]\right| =$$

$$= \frac{1}{2}\left|2[\vec{m}, \vec{m}] + [\vec{m}, \vec{n}] + 6[\vec{n}, \vec{m}] + 3[\vec{n}, \vec{n}]\right| =$$

$$= \frac{1}{2}\left|-5[\vec{m}, \vec{n}]\right| = \frac{5}{2}|\vec{m}||\vec{n}|\sin\frac{\pi}{6} = \frac{5}{2}\cdot 4\cdot 1\cdot\frac{1}{2} = 5. \quad \bullet$$

2. *Finding the linear velocity of the rotating body.*

Let the firm body rotates with angular velocity $\vec{\omega}$ around the motionless point O. At any moment the vector $\vec{\omega}$ coincides with the direction of the instantaneous axis of rotation l (Fig. 2.34).

The instantaneous linear velocity $\vec{v}$ of the arbitrary point M is defined by a vector

$$\vec{v} = [\vec{\omega}, \vec{r}].$$

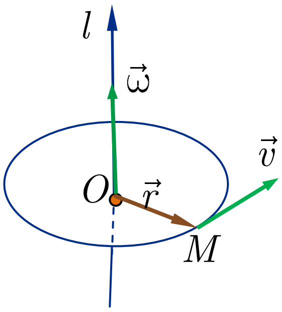

Fig. 2.34

3. *Finding the moment of the force about a point.*

Let $\vec{F}$ be a vector of force with the initial point at a point A (Fig. 2.35).

The moment $\vec{M}_O$ *of the force* $\vec{F}$ *about a point* O is called the vector

$$\vec{M}_O = [\overrightarrow{OA}, \vec{F}] = [\vec{r}, \vec{F}].$$

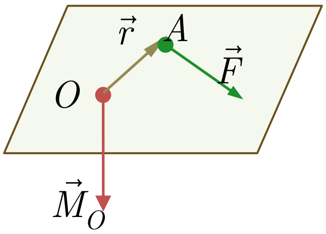

Fig. 2.35

When dot product and cross product are used together, the cross product takes precedence: $(\vec{a}[\vec{b}, \vec{c}])$. Because the dot product returns a scalar, $[(\vec{a}, \vec{b}), \vec{c}]$ is undefined since you cannot take the cross product of a scalar and a vector. The operation $(\vec{a}[\vec{b}, \vec{c}])$ is known as the *triple scalar product.*

2.2.3. Triple Scalar Product of Three Vectors

Definition of Triple Scalar Product

In space $\mathbb{R}^3$ each triplet of non-coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, attached to one point, represents the parallelepiped (Fig. 2.36), formed by these vectors as its edges.

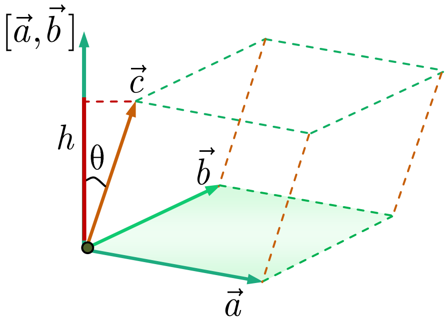

Fig. 2.36

We will consider this parallelepiped to be oriented, if the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ form oriented triplet.

Definition 2.9.

The triple scalar (vector-scalar) product of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is called the number which equals to the scalar product of the cross product of the vectors $\vec{a}$ and $\vec{b}$ by the vector $\vec{c}$:

$$([\vec{a},\vec{b}],\vec{c}) = (\vec{a},\vec{b},\vec{c}).$$

Basic Properties of Triple Scalar Product

Let us consider the main properties of the triple scalar product.

Property 1

In the triple scalar product we can change places of the signs of the cross product and the scalar product of the vectors:

$$([\vec{a},\vec{b}],\vec{c}) = (\vec{a},[\vec{b},\vec{c}]).$$

Property 2

The triple scalar product does not change in result of cyclic change between the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$:

$$(\vec{a},\vec{b},\vec{c}) = (\vec{c},\vec{a},\vec{b}) = (\vec{b},\vec{c},\vec{a}).$$

Property 3

If in the triple scalar product we change places of any two vectors then the sign of this product changes on the opposite:

$$(\vec{a},\vec{b},\vec{c}) = -(\vec{b},\vec{a},\vec{c}) = -(\vec{c},\vec{b},\vec{a}) = -(\vec{a},\vec{c},\vec{b}).$$

Property 4

The triple scalar product is linear above each multiplier, for instance,

$$(\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2,\vec{b},\vec{c}) = \alpha_1(\vec{a}_1,\vec{b},\vec{c}) + \alpha_2(\vec{a}_2,\vec{b},\vec{c}).$$

Property 5

The modulus of a triple scalar product $(\vec{a},\vec{b},\vec{c})$ is equal to the volume of the oriented parallelepiped, which is formed by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$:

$$\left|(\vec{a},\vec{b},\vec{c})\right| = V_{\text{par-d}}.$$

Property 6

If a triple scalar product $(\vec{a},\vec{b},\vec{c})$ is positive then the vectors form the right triplet, but when a triple scalar product of these vectors is negative then the vectors form the left triplet.

Property 7

The triple scalar product of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is equal to zero if and only if the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar:

$$(\vec{a},\vec{b},\vec{c}) = 0 \Leftrightarrow \text{the vectors } \vec{a},\vec{b},\vec{c} \text{ are coplanar}.$$

Applications of Triple Scalar Product

Now we will look at some ways in which the scalar product can be used.

1. *Calculating the volume of the parallelepiped* V_{par} *and of the triangle pyramid* V_{pir}.

The volume of the parallelepiped with $\vec{a}$, $\vec{b}$ and $\vec{c}$ as three of its edges is given by:

$$V_{\text{par}} = \left|(\vec{a},\vec{b},\vec{c})\right|.$$

The volume of the triangle pyramid built on the non-coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ we can find by the formula

$$V_{\text{pir}} = \frac{1}{6}\left|(\vec{a},\vec{b},\vec{c})\right|.$$

The height of the parallelepiped and of the triangle pyramid on the plane formed by the vectors $\vec{a}$ and $\vec{b}$ we find by the formula:

$$h = \left|\text{pr}_{[\vec{a},\vec{b}]}\vec{c}\right| = \frac{\left|(\vec{a},\vec{b},\vec{c})\right|}{\left|[\vec{a},\vec{b}]\right|} = \left|([\vec{a},\vec{b}]^0,\vec{c})\right|.$$

2. *Testing the coplanar condition and the orientation of three vectors.*

For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ we have;

if $(\vec{a},\vec{b},\vec{c}) > 0$ then vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ form the right triplet;

if $(\vec{a},\vec{b},\vec{c}) < 0$ then vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ form the left triplet;

if $(\vec{a},\vec{b},\vec{c}) = 0$ then vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar.

Remark

The triple scalar product is not the only useful way to multiply three vectors. An operation such as $[\vec{a},[\vec{b},\vec{c}]]$ *(called the "triple vector product") comes in very handy when you're dealing with certain problems involving angular momentum and centripetal acceleration.*

Unlike the triple scalar product, which produces a scalar result (since the second operation is a dot product), the triple vector product yields a vector result (since both operations are cross products).

The triple vector product (Fig.2.37) *can be expressed in terms of scalar products by the formula:*

$$[\vec{a},[\vec{b},\vec{c}]] = \vec{b}(\vec{a},\vec{c}) - \vec{c}(\vec{a},\vec{b})$$

*(you can remember it as the "*BAC *minus* CAB*" rule so long as you remember to write the members of the triple vector product in the correct sequence* $\vec{a},\vec{b},\vec{c}$ *).*

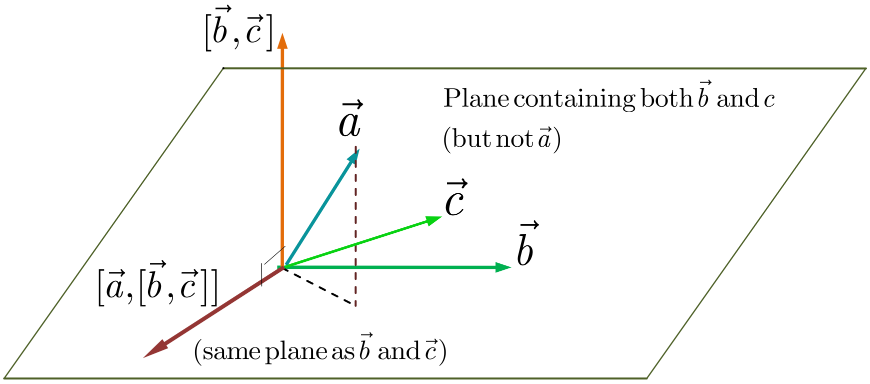

Fig.2.37