

- Cartesian Coordinate System
- Dot Product, Cross Product of Two Vectors in Cartesian Basis
- Triple Scalar Product of Three Vectors in Cartesian Basis

We have defined vectors, and operations with them, geometrically, in terms of vectors' magnitude and direction. However, for a variety of problems in physics and engineering, it is immensely useful to treat vectors algebraically. This requires the introduction of a Coordinate System which is familiar to you from the school math course.

2.2.4. Cartesian Coordinate System

The *Cartesian (Rectangular) Coordinate System* is the conditions, as you know, that define the position of points, vectors on the line, on the plane, in space by means of numbers.

Cartesian Coordinate System on a Line

Let us fix on the line the point O , we will call it *the origin*, then we choose *nonzero unit vector* \vec{i} as a basis on the line (Fig. 2.38).

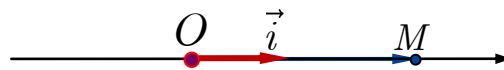


Fig. 2.38

Consider an arbitrary point M on the axis and decompose the vector \overrightarrow{OM} in the basis \vec{i} ($\overrightarrow{OM} \parallel \vec{i}$):

$$\overrightarrow{OM} = x\vec{i}.$$

Definition 2.10.

The coefficient x is called *the Cartesian coordinate* of the vector \overrightarrow{OM} in the basis $\{\vec{i}\}$.

By the property of the projection of a vector onto axis we have:

$$x = \text{pr}_{\vec{i}} \overrightarrow{OM}.$$

Cartesian Coordinate System in 2- dimension

A Cartesian Coordinate System on a plane is constructed as we know by selecting two axes that are intersecting at right angles, and the point of intersection we label as O .

Let \vec{i} and \vec{j} are two unit mutually perpendicular vectors on these axes. They form *the standard orthogonal basis* on the plane (Fig. 2.39):

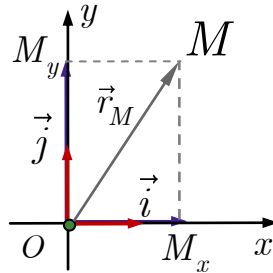


Fig. 2.39

In this case we say that we have *the Cartesian coordinate system in 2-dimension*. The point O we call *the origin of the Coordinate system*. The axes with direction vectors \vec{i} and \vec{j} are called the x -axis and y -axis respectively. Such plane is the coordinate plane Oxy .

Let us consider an arbitrary point M on the plane. We may express its radius-vector $\vec{r}_M = \overrightarrow{OM}$ in very handy and compact way (see Fig. 2.39) in the basis \vec{i}, \vec{j} as

$$\overrightarrow{OM} = \overrightarrow{OM_x} + \overrightarrow{OM_y} = x\vec{i} + y\vec{j},$$

$$\overrightarrow{OM_x} \parallel \vec{i}, \quad \overrightarrow{OM_y} \parallel \vec{j},$$

where the magnitudes of the vector components (x, y) tell us how many unit steps to take in each direction (\vec{i} and \vec{j}) to get from the beginning to the end of vector \overrightarrow{OM} . So, on the plane two numbers are used to specify a location. (The fact that we use two numbers to describe the location is the reason it's called *two-dimensional space*).

Definition 2.11.

The coefficients x and y are called *the Cartesian coordinates* of the vector \overrightarrow{OM} in the Cartesian coordinate system in 2-dimension in the orthogonal basis \vec{i}, \vec{j} .

We may write in this case vector \overrightarrow{OM} as a matrix-column:

$$\overrightarrow{OM} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

We have (see properties of the projection):

$$x = \text{pr}_{Ox} \overrightarrow{OM} = \text{pr}_{\vec{i}} \overrightarrow{OM} = \overrightarrow{OM} \cdot \vec{i},$$

$$y = \text{pr}_{Oy} \overrightarrow{OM} = \text{pr}_{\vec{j}} \overrightarrow{OM} = \overrightarrow{OM} \cdot \vec{j},$$

$$|\overrightarrow{OM}|^2 = x^2 + y^2.$$

To understand how this works, take a look at vector $\vec{r}_M = \overrightarrow{OM}$ and the light sources and shadows in Fig. 2.40.

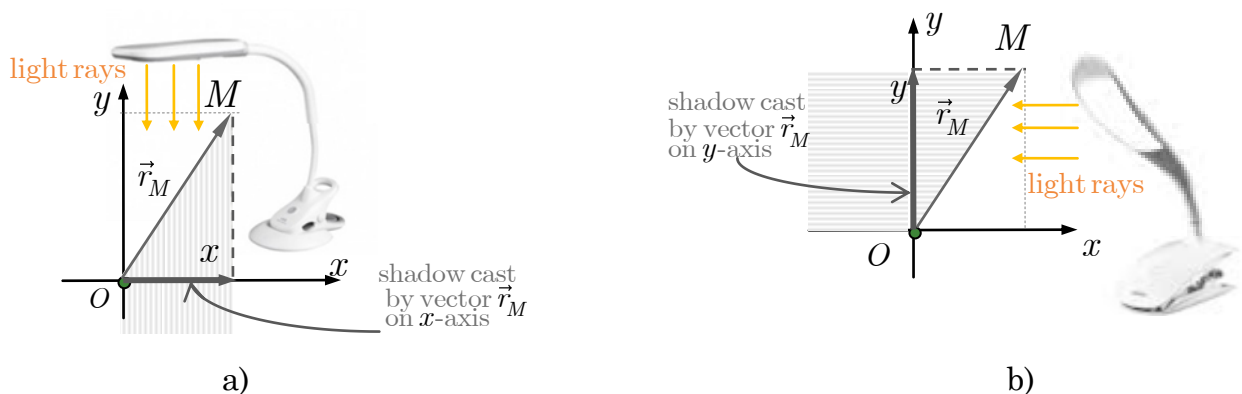


Fig. 2.40

As you can see in Fig. 2.40 a), the direction of the light that produces the shadow on the x -axis is parallel to the y -axis (actually the light is moving in the negative y -direction), which in this case is the same as saying that the direction of the light is perpendicular to the x -axis.

Likewise, in Fig. 2.40 b), the direction of the light that produces the shadow on the y -axis is parallel to the x -axis (even though and in this case the light is moving in the negative x -direction), which is of course perpendicular to the y -axis.

Cartesian Coordinate System in 3- dimension

In 3-dimension, we require three axes to establish a coordinate system. First choose a point O called the origin, then choose three mutually perpendicular lines through O , called the x , y , and z axes.

Let \vec{i} , \vec{j} and \vec{k} are three unit mutually perpendicular vectors on the axes. They are non-coplanar and, hence, form the orthogonal basis in space (Fig. 2.41).

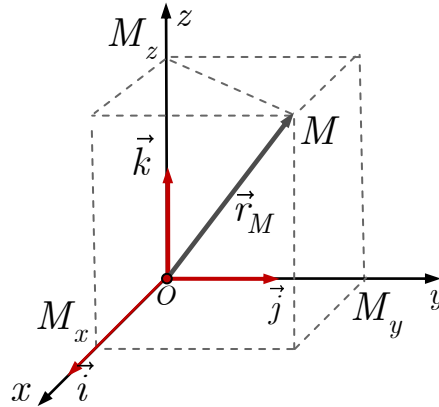


Fig. 2.41

The radius-vector $\vec{r}_M = \overrightarrow{OM}$ of the arbitrary point M in the basis $\vec{i}, \vec{j}, \vec{k}$ we may represent (see Fig. 2.41) as

$$\overrightarrow{OM} = \overrightarrow{OM_x} + \overrightarrow{OM_y} + \overrightarrow{OM_z} = x\vec{i} + y\vec{j} + z\vec{k}.$$

Definition 2.12.

The coefficients x , y and z are called *the Cartesian coordinates* of the vector \overrightarrow{OM} in the Cartesian coordinate system in space in the orthogonal basis $\{\vec{i}, \vec{j}, \vec{k}\}$:

$$\overrightarrow{OM} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$



Let's note, that

$$\begin{aligned} x &= \text{pr}_{Ox} \overrightarrow{OM} = \text{pr}_{\vec{i}} \overrightarrow{OM} = \overrightarrow{OM} \cdot \vec{i}, \\ y &= \text{pr}_{Oy} \overrightarrow{OM} = \text{pr}_{\vec{j}} \overrightarrow{OM} = \overrightarrow{OM} \cdot \vec{j}, \\ z &= \text{pr}_{Oz} \overrightarrow{OM} = \text{pr}_{\vec{k}} \overrightarrow{OM} = \overrightarrow{OM} \cdot \vec{k}, \end{aligned}$$

$$|\overrightarrow{OM}|^2 = x^2 + y^2 + z^2.$$

In the basis $\{\vec{i}, \vec{j}, \vec{k}\}$ we have: if

$$\vec{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = x_1 \vec{i} + y_1 \vec{j} + z_1 \vec{k} \quad \text{and} \quad \vec{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_2 \vec{i} + y_2 \vec{j} + z_2 \vec{k},$$

then in terms of coordinates

$$\vec{a} = \vec{b} \Leftrightarrow \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x_1 = x_2 \\ y_1 = y_2 \\ z_1 = z_2 \end{pmatrix}; \quad \lambda \vec{a} = \begin{pmatrix} \lambda x_1 \\ \lambda y_1 \\ \lambda z_1 \end{pmatrix}$$

$$\vec{a} + \vec{b} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}; \quad \vec{a} \parallel \vec{b} \Leftrightarrow \frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}.$$

Direction Cosines of Vector

Let vector \vec{a} has the coordinates $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ in the basis $\{\vec{i}, \vec{j}, \vec{k}\}$ (Fig. 2.42):

$$\vec{a} = x_1 \vec{i} + y_1 \vec{j} + z_1 \vec{k}.$$

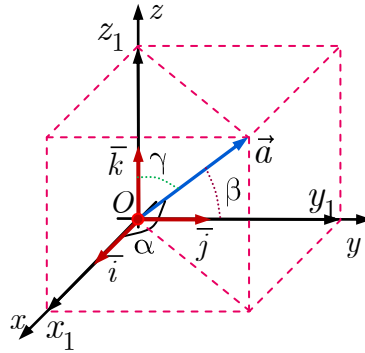


Fig. 2.42

By the theorem about the projection of a vector onto an axis we get (see Fig. 2.42):

$$x_1 = \text{pr}_{Ox} \vec{a} = |\vec{a}| \cos \alpha,$$

$$y_1 = \text{pr}_{Oy} \vec{a} = |\vec{a}| \cos \beta, \quad (2.2)$$

$$z_1 = \text{pr}_{Oz} \vec{a} = |\vec{a}| \cos \gamma,$$

here

$$\alpha = (\vec{a}, Ox), \quad \beta = (\vec{a}, Oy), \quad \gamma = (\vec{a}, Oz).$$



$\cos \alpha$, $\cos \beta$, $\cos \gamma$ are called *the Direction Cosines* of the vector \vec{a} .

From (2.2) it follows that

$$\cos \alpha = \frac{x_1}{|\vec{a}|}, \quad \cos \beta = \frac{y_1}{|\vec{a}|}, \quad \cos \gamma = \frac{z_1}{|\vec{a}|},$$

and the unit vector

$$\vec{a}^0 = \frac{\vec{a}}{|\vec{a}|}$$

has the coordinates

$$\vec{a}^0 = \begin{pmatrix} \frac{x_1}{|\vec{a}|} \\ \frac{y_1}{|\vec{a}|} \\ \frac{z_1}{|\vec{a}|} \end{pmatrix} = \begin{pmatrix} \cos \alpha \\ \cos \beta \\ \cos \gamma \end{pmatrix}.$$

N.B. Hence, the coordinates of any unit vector are equal to the direction cosines.

So, we have obtained the basic relation between the direction cosines of a vector

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

Remark



In two dimensions, the set of all unit vectors \vec{u} may be parameterized in terms of their angle relative to the horizontal axis (Fig. 2.43):

$$\vec{u} = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}.$$

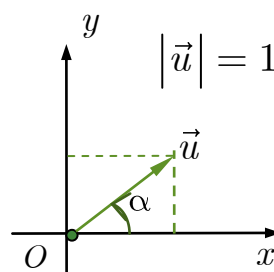


Fig.2.43

Scalar Product of Vectors in Cartesian Basis

Let's consider two vectors \vec{a} and \vec{b} in Cartesian orthogonal basis $\{\vec{i}, \vec{j}, \vec{k}\}$:

$$\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k},$$

$$\vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}.$$

By the properties of a scalar product of two vectors we get:

$$\begin{aligned} (\vec{a}, \vec{b}) &= x_1x_2(\vec{i}, \vec{i}) + y_1y_2(\vec{j}, \vec{j}) + z_1z_2(\vec{k}, \vec{k}) + \\ &+ (x_1y_2 + y_1x_2)(\vec{i}, \vec{j}) + (x_1z_2 + z_1x_2)(\vec{i}, \vec{k}) + (y_1z_2 + z_1y_2)(\vec{j}, \vec{k}). \end{aligned}$$

Due to the unit vectors $\vec{i}, \vec{j}, \vec{k}$ are the mutually perpendicular vectors we obtain (Fig. 2.44)

$(,)$	\vec{i}	\vec{j}	\vec{k}
\vec{i}	1	0	0
\vec{j}	0	1	0
\vec{k}	0	0	1

Fig. 2.44

and thus, we have

$$(\vec{a}, \vec{b}) = x_1x_2 + y_1y_2 + z_1z_2.$$

On occasions you may see this form referred to as the *inner product* of the vectors $\vec{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$. In the context of vectors this simply means the sum of the products of the corresponding vector coordinates.

This is nothing but the product of the matrix

$$\vec{a}^T = (x_1 \quad y_1 \quad z_1)^T$$

and the matrix

$$\vec{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}.$$

i.e.

$$(\vec{a}, \vec{b}) = \vec{a}^T \cdot \vec{b}$$

Since $|\vec{a}| = \sqrt{(\vec{a}, \vec{a})}$ the length of the vector \vec{a} is the square root of the sum of the squares of the coordinates of the vector



N.B.

$$|\vec{a}| = \sqrt{x_1^2 + y_1^2 + z_1^2},$$

$$\cos(\vec{a}, \vec{b}) = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2} \cdot \sqrt{x_2^2 + y_2^2 + z_2^2}},$$

$$\text{pr}_{\vec{b}}\vec{a} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_2^2 + y_2^2 + z_2^2}}.$$

The orthogonality condition of the vectors \vec{a} and \vec{b} in coordinates has the form

$$x_1x_2 + y_1y_2 + z_1z_2 = 0.$$

Cross Product of Vectors in Cartesian Basis

Given the vectors

$$\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$$

and

$$\vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$$

in a Cartesian basis. By the properties of the cross product of the unit mutually perpendicular vectors \vec{i} , \vec{j} , \vec{k} we have (Fig. 2.45)

$[,]$	\vec{i}	\vec{j}	\vec{k}
\vec{i}	$\vec{0}$	\vec{k}	$-\vec{j}$
\vec{j}	$-\vec{k}$	$\vec{0}$	\vec{i}
\vec{k}	\vec{j}	$-\vec{i}$	$\vec{0}$

Fig. 2.45

Let us find the cross product of the given vectors \vec{a} and \vec{b} using the table above:

$$\begin{aligned}
 [\vec{a}, \vec{b}] &= [(x_1\vec{i} + y_1\vec{j} + z_1\vec{k}), (x_2\vec{i} + y_2\vec{j} + z_2\vec{k})] = \\
 &= x_1x_2[\vec{i}, \vec{i}] + y_1y_2[\vec{j}, \vec{j}] + z_1z_2[\vec{k}, \vec{k}] + \\
 &+ x_1y_2[\vec{i}, \vec{j}] + x_1z_2[\vec{i}, \vec{k}] + (y_1x_2)[\vec{j}, \vec{i}] + \\
 &+ (y_1z_2)[\vec{j}, \vec{k}] + (z_1x_2)[\vec{k}, \vec{i}] + (z_1y_2)[\vec{k}, \vec{j}] = \\
 &= (y_1z_2 - z_1y_2)\vec{i} + (z_1x_2 - x_1z_2)\vec{j} + (x_1y_2 - y_1x_2)\vec{k}.
 \end{aligned}$$

Remembering the determinants of the second order and the definition of the determinant we obtain:

$$[\vec{a}, \vec{b}] = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \vec{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \vec{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}.$$

Therefore,

$$[\vec{a}, \vec{b}] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}.$$

Example

Let's calculate the area of the triangle ABC which is built on the vectors $\overrightarrow{AB}(2; -2; -3)$ and $\overrightarrow{AC}(4; 0; 6)$.

○ As we know,

$$S_{ABC} = \frac{1}{2} |[\overrightarrow{AB}, \overrightarrow{AC}]|.$$

Vectors are given in the coordinate form:

$$\overrightarrow{AB}(2; -2; -3), \quad \overrightarrow{AC}(4; 0; 6).$$

And we get

$$[\overrightarrow{AB}, \overrightarrow{AC}] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -3 \\ 4 & 0 & 6 \end{vmatrix} = -12\vec{i} - 24\vec{j} + 8\vec{k}.$$

Thus, we have

$$S_{ABC} = \frac{1}{2} |[\overrightarrow{AB}, \overrightarrow{AC}]| = \frac{1}{2} \sqrt{(-12)^2 + (-24)^2 + 8^2} = 2\sqrt{9 + 36 + 4} = 14. \bullet$$

Example

Let's find the unit vector \vec{e} that is orthogonal to the vectors $\vec{p}(3; 1; 0)$, $\vec{q}(-1; 2; 1)$ and forms the acute angle with axis Oy .

○ Vector \vec{e} is orthogonal to both vectors \vec{p} , \vec{q} ($\vec{e} \perp \vec{p}$, $\vec{e} \perp \vec{q}$). This condition follows that vectors \vec{e} and $[\vec{p}, \vec{q}]$ are collinear:

$$\vec{e} \parallel [\vec{p}, \vec{q}].$$

It means that coordinates of these vectors are proportional:

$$\vec{e} = \lambda [\vec{p}, \vec{q}],$$

$$|\lambda| = \frac{|\vec{e}|}{\|[\vec{p}, \vec{q}]\|} = \frac{1}{\|[\vec{p}, \vec{q}]\|}.$$

Let us find the coordinates of the vector $[\vec{p}, \vec{q}]$:

$$[\vec{p}, \vec{q}] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 0 \\ -1 & 2 & 1 \end{vmatrix} = \vec{i} - 3\vec{j} + 7\vec{k}.$$

So, the vector \vec{e} has the coordinates

$$\vec{e} = \lambda \vec{i} - 3\lambda \vec{j} + 7\lambda \vec{k}$$

The length of the vector $[\vec{p}, \vec{q}]$ is

$$\|[\vec{p}, \vec{q}]\| = \sqrt{1^2 + (-3)^2 + 7^2} = \sqrt{59}$$

Therefore,

$$|\lambda| = \frac{1}{\sqrt{59}} \Rightarrow \lambda = \pm \frac{1}{\sqrt{59}}.$$

According to the condition of the problem the angle between vector \vec{e} and axis Oy is acute:

$$0 < \beta = (\vec{e}, Oy) < \frac{\pi}{2} \Rightarrow \cos \beta > 0,$$

consequently, the second coordinate of vector \vec{e} have to be positive.

This condition gives inequality

$$-3\lambda > 0 \Rightarrow \lambda < 0.$$

And we take

$$\lambda = -\frac{1}{\sqrt{59}}.$$

Thus, vector \vec{e} has the coordinates

$$\vec{e} = \left(\frac{-1}{\sqrt{59}}; \frac{3}{\sqrt{59}}; \frac{-7}{\sqrt{59}} \right)$$

Triple Scalar Product of Vectors in Cartesian Basis

Given vectors

$$\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k},$$

$$\vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k},$$

and

$$\vec{c} = x_3\vec{i} + y_3\vec{j} + z_3\vec{k}$$

Let us find the triple scalar product of these three vectors. We have

$$\begin{aligned}(\vec{a}, \vec{b}, \vec{c}) &= ([\vec{a}, \vec{b}], \vec{c}) = \\ &= \left(\left(\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \vec{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \vec{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \vec{k}, x_3\vec{i} + y_3\vec{j} + z_3\vec{k} \right) = \\ &= \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} x_3 - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} y_3 + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} z_3 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.\end{aligned}$$

Hence,

$$(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Example

Let's express the vector $\vec{d}(3;3;-1)$ by the vectors $\vec{p}(3;1;0)$, $\vec{q}(-1;2;1)$ and $\vec{r}(-1;0;2)$.

○ First, we have to prove that the vectors \vec{p} , \vec{q} , \vec{r} form the basis in 3-dimension, in another words we have to prove that these vectors are linearly independent, i.e. non-coplanar. It means that their triple scalar product doesn't equal zero.

Let's calculate triple scalar product of $\vec{p}, \vec{q}, \vec{r}$:

$$(\vec{p}, \vec{q}, \vec{r}) = \begin{vmatrix} 3 & 1 & 0 \\ -1 & 2 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 12 - 1 + 2 = 13 \neq 0.$$

It follows that triple scalar product of the vectors $\vec{p}, \vec{q}, \vec{r}$ doesn't equal zero as required. So the vectors $\vec{p}, \vec{q}, \vec{r}$ form the basis.

In the basis $\{\vec{p}, \vec{q}, \vec{r}\}$ vector \vec{d} has the coordinates x, y, z that are the expansion coefficients in this basis:

$$\vec{d} = x\vec{p} + y\vec{q} + z\vec{r}.$$

In coordinate form equality above takes the form

$$\begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} = x \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

And we obtain the linear system

$$\begin{cases} 3x - y - z = 3 \\ x + 2y = 3 \\ y + 2z = -1. \end{cases}$$

As we know we can solve this system using matrix method, Cramer's rule, Gauss' method (the determinant of the coefficient matrix of this system doesn't equal zero).

As a result of simple calculations we get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Thus,

$$\vec{d} = \vec{p} + \vec{q} - \vec{r}. \quad \bullet$$

2.3. Some Applications of Vector Algebra

(Main Simple Problems of Analytic Geometry)

- *Coordinates of Vector*
- *Distance between two points*
- *Division of the Line Segment in a given ratio*

Coordinates of Vector

Let us consider in a Cartesian coordinate system $Oxyz$ two points $A(x_1; y_1; z_1)$ and $B(x_2; y_2; z_2)$. We have (Fig. 2.46):

$$\overrightarrow{AB} = \vec{r}_B - \vec{r}_A.$$

As we know radius-vectors \vec{r}_A and \vec{r}_B have the coordinates of the points A and B respectively:

$$\vec{r}_A(x_1; y_1; z_1), \quad \vec{r}_B(x_2; y_2; z_2).$$

Thus we obtain by the properties of the coordinates:

$$\overrightarrow{AB} = \vec{r}_B - \vec{r}_A = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}.$$

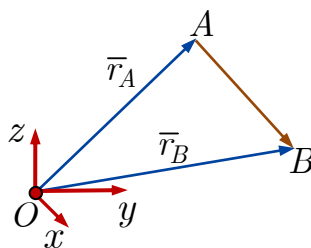


Fig. 2.46

Distance between Two Points

We are now prepared to introduce one of the most important formulas in geometry — *the distance formula*. This formula is used to compute the distance between two points. Since a vector is a directed line segment, it makes sense geometrically that the distance $d(A,B)$ between the two points $A(x_1; y_1; z_1)$ and $B(x_2; y_2; z_2)$ would be equal to the length of a vector \overrightarrow{AB} . Let's derive the distance formula. First, we will compute the coordinates of the vector \overrightarrow{AB} (Fig. 2.47):

$$\overrightarrow{AB} = \vec{r}_B - \vec{r}_A = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}.$$

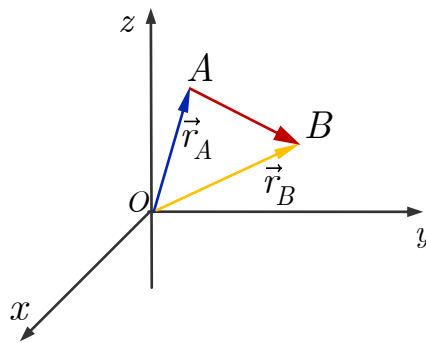


Fig. 2.47

So, we get

$$d(A,B) = \left| \overrightarrow{AB} \right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$

or in the matrix form

$$d(A,B) = \sqrt{(\vec{r}_B - \vec{r}_A)^T (\vec{r}_B - \vec{r}_A)}.$$

Division of the Line Segment in a Given Ratio

Finally we consider the problem of dividing a line segment in a given ratio. We wish to divide the line segment joining $A(x_1; y_1; z_1)$ and $B(x_2; y_2; z_2)$ internally in the ratio $\lambda > 0$ (Fig. 2.48).

We say that a point M divides the segment AB in a ratio λ , if the following equality is valid

$$|AM| = \lambda |MB|.$$

Let's consider the vector \overrightarrow{AM} and the vector \overrightarrow{MB} (see Fig. 2.48). The point $M(x; y; z)$ divides the segment AB in ratio λ if

$$\overrightarrow{AM} = \lambda \cdot \overrightarrow{MB}.$$

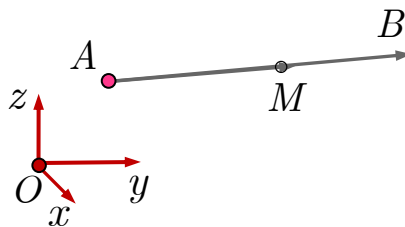


Fig. 2.48

But

$$\overrightarrow{AM} = \begin{pmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{pmatrix}, \quad \overrightarrow{MB} = \begin{pmatrix} x_2 - x \\ y_2 - y \\ z_2 - z \end{pmatrix},$$

therefore

$$\begin{pmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{pmatrix} = \lambda \cdot \begin{pmatrix} x_2 - x \\ y_2 - y \\ z_2 - z \end{pmatrix}.$$

We obtain the system

$$\begin{cases} x - x_1 = \lambda(x_2 - x), \\ y - y_1 = \lambda(y_2 - y), \\ z - z_1 = \lambda(z_2 - z), \end{cases}$$

whence

$$\begin{cases} (1 + \lambda)x = (x_1 + \lambda x_2), \\ (1 + \lambda)y = (y_1 + \lambda y_2), \\ (1 + \lambda)z = (z_1 + \lambda z_2). \end{cases}$$

Thus,

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda}; y = \frac{y_1 + \lambda y_2}{1 + \lambda}; z = \frac{z_1 + \lambda z_2}{1 + \lambda}.$$

Remark



1. If $\lambda = 0$, then the points A and M coincide (Fig. 2.49).
2. If $\lambda > 0$ then M lies inside AB (see Fig. 2.49).
3. If $\lambda < 0, \lambda \neq -1$ then the point M lies outside the segment AB (see Fig. 2.47).
4. If $\lambda = 1$ then the point M is a middle of the segment AB (see Fig. 2.49) and

$$x_M = \frac{x_1 + x_2}{2}; y_M = \frac{y_1 + y_2}{2}; z_M = \frac{z_1 + z_2}{2}.$$

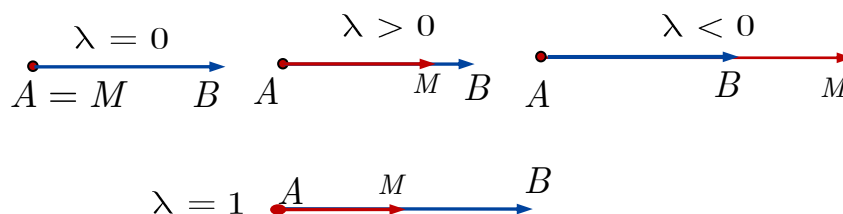


Fig. 2.49

Questions to Chapter 2

I.

1. What refer to the vectors?
2. What is the unit vector?
3. What are the collinear vectors?
4. State the linear operations above vectors.
5. What is the projection of a vector onto the axis?
6. What properties of the projection do you know?
7. Give the examples of the linear space.
8. What are the coordinates of a vector in the space?
9. What is the basis on the line, on the plane, in space?
10. What is the scalar product of two vectors?
11. State the properties of the scalar product.
12. What problems may be solved by the scalar product?
13. What is the cross product of two vectors?
14. State the properties of the cross product.
15. What problems may be solved by the cross product?
16. What is the triple scalar product of three vectors?
17. State the properties of the triple scalar product.
18. What problems may be solved by the triple scalar product?
19. What are the coordinates of the unit vector?
20. State the properties of the linearly independent systems of vectors.
21. State the orthogonality condition of two vectors.

II.

Consider three vectors

$$\vec{a}(3;4;6), \quad \vec{b}(4;1;1), \quad \vec{c}(2;0;3).$$

1. Calculate

$$\Delta = \begin{vmatrix} 3 & 4 & 6 \\ 4 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix}.$$

- What can you tell about Δ ?
- State the geometric sense of Δ .

2. Calculate

$$\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 6 \\ 4 & 1 & 1 \end{vmatrix}.$$

- What is \vec{w} ?
 - How the vector \vec{w} relates to vectors \vec{a} and \vec{b} ?
- Calculate (\vec{w}, \vec{a}) , (\vec{w}, \vec{b}) .
 - Calculate $|\vec{w}|$. State the geometric sense of $|\vec{w}|$.
 - Find the direction cosines of the vector \vec{w} .
 - Calculate (\vec{w}, \vec{c}) . How Δ relates to (\vec{w}, \vec{c}) ?
 - Find the unit vector in the direction of vector \vec{w} .
 - Calculate (\vec{w}^0, \vec{c}) . State the geometric sense of $|\vec{w}^0, \vec{c}|$.