Chapter 3

Fundamentals of Grassman Algebra

3.1. Complex Numbers

- *Axiomatic definition of Complex Number (Standard Form)*
- *Cartesian Form of Complex Number*
- *Geometrical Representation of Complex Numbers (Vector Form)*
- *Matrix Form of Complex Number*
- *Polar Form of Complex Number (Power and Root of Complex Number)*

The theory of complex numbers is one of the most beautiful chapters of mathematics. The way to their discovery was not an easy one, as evidenced by even the terminology: “complex” numbers were called “impossible”, “imaginary” numbers, even the modern name (associated with the English adjective – complex) caused the feeling that they were difficult to understand. Fortunately, this is no longer the case.

The earliest traces of imaginary numbers find themselves in Italy. In 1545 an Italian mathematician Gerolamo Cardano published a book entitled “The Great Art”, in which he described an algebraic method of solving cubic equations. In solutions of cubic equations Cardano obtained square roots of negative numbers.

Gerolamo Cardano

The problem of solving quadratic equations like $x^2+1=0$ was considered meaningless while performing calculations with expressions that contain a square root of a negative number can yield quite understandable results (a real cubic equation has a real solution).

Therefore, these roots began to apply in mathematics. They were called imaginary numbers — thus, they seemed to have gotten the right to illegal existence.

Felix Klein

Felix Klein remarked on the further development of complex numbers: "Independently and even against the will of one or another mathematician, imaginary numbers appear again and again in calculations,but only gradually, as they prove to be beneficial in their application, they receive everything more and more distribution".

Of course, mathematicians did it not with a light heart, imaginary numbers kept a mystical color for a long time. *Gottfried Leibniz* called the imaginary numbers “a fine and wonderful refuge of the divine spirit – almost an amphibian between being and non-being”.

Gottfried Leibniz

Let's try to describe the complex numbers in a relatively simple way.

The real numbers are known to be represented by the points of the numeric line: positive numbers are to the right, and negative numbers are to the left of the origin. Points are geometric objects, and numbers are algebraic ones. If the points on the line are numbers, then the simplest operations over numbers (addition and multiplication) must have geometric meaning.

The key to understanding this content is in the idea of a transformation. For example, $x+2$ this is a shift of x to the right (transfer); $x\cdot 2$— extension (scale), $x\cdot(-1)$ translates a point x into a point $-x$ (we can understand it as a symmetry about the origin) (Fig. 3.1).

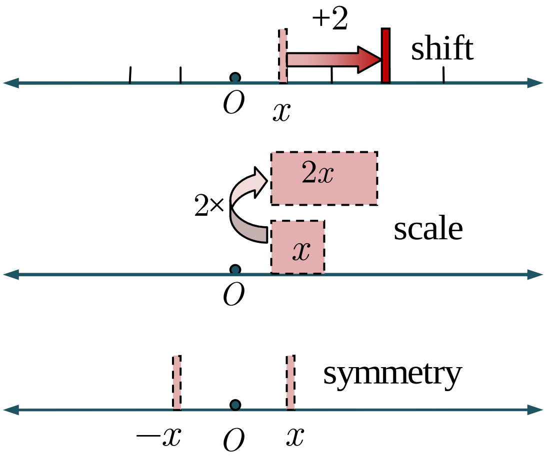

Fig. 3.1

The number (-1) corresponds to the central symmetry with respect to zero, that is, a 180 degrees rotation. When we perform this operation twice in a row, we get back the original point, just as the product of (-1) with itself is $(+1)$. The square of (-1) is $(+1)$.

What does the symbol $\sqrt{-1}$ *mean?*

To find $\sqrt{-1}$, you need to find a transformation, which, when applied it twice, gives a 180 degrees rotation. Consequently, $\sqrt{-1}$ must correspond to a rotation of 90 degrees (Fig. 3.2). The application of two turns of 90 degrees gives a 180 degrees rotation, that is multiplication by (-1).

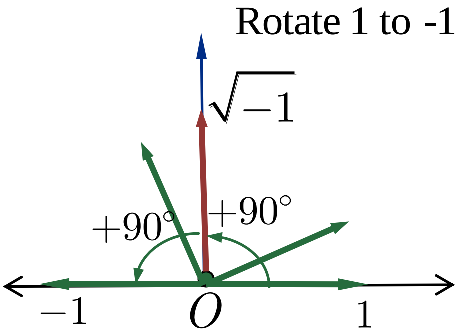

Fig. 3.2

The image of 1, when radius-vector is rotated by 90 degrees, does not lie on the numeric line, and therefore $\sqrt{-1}$ is decided to be a point, that lies on a plane.

N.B. *The idea is simple and beautiful:*

consider the point on the plane as a number.

3.1.1. Axiomatic Definition of Complex Number (Standard Model)

We'll give a definition of a Complex Number, based on the fact that real numbers $\mathbb{R}$ are entered and two main operations — addition and multiplication with well-known properties — are performed for them.

Definition 3.1.

Complex Number is called an ordered pair of real numbers

$$(x, y),\ x, y \in \mathbb{R},$$

which satisfies the following axioms:

1. *axiom of identity*:

$$(x_1, y_1) = (x_2, y_2) \Leftrightarrow (x_1 = x_2) \wedge (y_1 = y_2);$$

2. *axiom of addition*:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2);$$

3. *axiom of multiplication*:

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1);$$

4. *axiom of relation*:

$$(x, 0) \equiv x.$$

The sum of complex numbers is commutative, associative, with neutral element (called *zero*) $(0,0)$ and opposite $(-x,-y)$ of (x,y). The multiplication of complex numbers is commutative, associative, with unit $(1,0)$, inverse element of any nonzero complex number, and with the distributive property.

For numbers $(x,0)$ all operations for real numbers and their properties are valid. This complex number is identified with a real number.

From the given system of axioms we obtain such corollaries:

Corollary 1

$$\forall \alpha \in \mathbb{R}: \quad \alpha \cdot (x, y) = (\alpha, 0) \cdot (x, y) = (\alpha x, \alpha y).$$

The number

$$(1, 0) \equiv 1$$

plays the role of unity, or one, i.e. for any complex number (x, y) *we have:*

$$1 \cdot (x, y) = (1, 0)(x, y) = (x, y)(1, 0) = (x, y).$$

Corollary 2

$$(x_1, y_1) - (x_2, y_2) = (x_1, y_1) + (-1)(x_2, y_2) =$$

$$= (x_1, y_1) + (-x_2, -y_2) = (x_1 - x_2, y_1 - y_2)$$

Let us consider the complex number $(0, 1)$ and denote it by i (j for Electrical Engineers):

$$i = (0, 1).$$

We have:

$$(0, 1)(0, 1) = (-1, 0) \equiv -1,$$

i.e.

$$i^2 = -1.$$

The complex number i is called *the imaginary unit* (i – the first letter of the word *imaginary*).

Let’s find the product of real number y and imaginary unit i:

$$iy = (0, 1)(y, 0) = (0, y).$$

3.1.2. Cartesian Complex Numbers

For algebraic manipulation it is not convenient to represent a complex number as an ordered pair. For this reason another form of writing is preferred.

Any complex number may be written as

$$(x,y)=(x,0)+(0,y)=(x,0)+i(y,0)=x+iy\,.$$

Sometimes it is convenient to use a single letter, such as z, to denote a complex number. Thus we might write expression

$$z=x+iy$$

which is called *a Cartesian Complex Number* (*the algebraic form of the complex number*).

The numbers x and y are called, respectively, *the Real Part* and *the Imaginary Part* of a complex number z. They are symbolized as

$$x=\operatorname{Re} z\in\mathbb{R},\qquad y=\operatorname{Im} z\in\mathbb{R}\,.$$

A complex number $x+0i$ is identified with a real number x.

Two complex numbers

$$z_1=x_1+iy_1 \quad\text{and}\quad z_2=x_2+iy_2$$

are defined to be *equal*, if their real parts are equal and their imaginary parts are equal too:

$$z_1=z_2 \Leftrightarrow \begin{cases} x_1=x_2, \\ y_1=y_2. \end{cases}$$

Remark

N.B. *Unlike the real numbers, there is no size ordering for the complex numbers. Thus, the order symbols <, ≤, >, and ≥ are not used with complex numbers.*

We are now about to show basic operations for Cartesian complex numbers. We'll start with addition and subtraction.

The sum of two Cartesian Complex Numbers

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2$$

is called the complex number, defined by the equality

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

The difference of Complex Numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is the complex number

$$z_1 - z_2 = z_1 + (-z_2) = (x_1 - x_2) + i(y_1 - y_2).$$

The easiest way to think of adding and subtracting Complex Numbers is to think of each Complex Number as a polynomial and do the addition and subtraction in the same way that we add or subtract polynomials.

Next let's take a look at multiplication.

The product of the Real Number α *and the Complex Number* $z = x + iy$ is called *the Complex Number*

$$\alpha z = \alpha x + i\alpha y.$$

The product of two Complex Numbers

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2$$

is called the complex number

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

We multiply given complex numbers as polynomials

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2$$

and consider that

$$i^2 = -1:$$

$$z_1 z_2 = x_1 x_2 + ix_1 y_2 + iy_1 x_2 - y_1 y_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2).$$

We leave it as an exercise to verify the following rules of complex arithmetic for the interested reader:

1) $z_1 + z_2 = z_2 + z_1$;

2) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$;

3) $z + 0 = z$ (*the complex number* $0 = 0{+}0i$ *is an additive identity*);

4) $z_1 z_2 = z_2 z_1$;

5) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$;

6) $1 \cdot z = z$ (*the complex number* $1 = 1{+}0i$ *is a multiplicative identity*);

7) $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$.

Remark

Addition and multiplication of Complex Numbers are achieved by assuming all quantities involved are real and then using $i^2 = -1$ *to simplify.*

Example

Given

$$z_1 = 2 + i\,,\ z_2 = 7 - 3i\,.$$

Let's find

$$z_1 + z_2\,,\ \ z_1 - z_2\,,\ \ z_1 z_2\,.$$

❍ According to the rules of addition and multiplication of complex numbers we get:

$$z_1 + z_2 = (2 + i) + (7 - 3i) = (2 + 7) + (1 - 3)i = 9 - 2i\,,$$

$$z_1 - z_2 = (2 + i) - (7 - 3i) = (2 - 7) + (1 - (-3))i = -5 + 4i\,,$$

$$z_1 \cdot z_2 = (2 + i) \cdot (7 - 3i) = (2 \cdot 7 - 1 \cdot (-3)) + (1 \cdot 7 + 2 \cdot (-3))i = 17 + i\,.$$ ●

Now that we have defined addition, subtraction, and multiplication of complex numbers, it is possible to do all the standard linear algebra calculations over complex numbers — add, subtract, multiply matrices with complex entries, and multiply a matrix by a complex number. Without going into detail, we note that the matrix operations and terminology discussed in Chapter 1 carry over without change to matrices with complex elements.

If

$$\mathrm{A} = \begin{pmatrix} 1 & i \\ 3+i & -2 \end{pmatrix} \quad \text{and} \quad \mathrm{B} = \begin{pmatrix} -i & 2-i \\ 3+2i & 1 \end{pmatrix}$$

then

$$\mathrm{A}+\mathrm{B} = \begin{pmatrix} 1-i & 2 \\ 6+3i & -1 \end{pmatrix}, \qquad i\mathrm{B} = \begin{pmatrix} 1 & 1+2i \\ -2+3i & i \end{pmatrix},$$

$$\mathrm{A}\cdot\mathrm{B} = \begin{pmatrix} 1 & i \\ 3+i & -2 \end{pmatrix} \cdot \begin{pmatrix} -i & 2-i \\ 3+2i & 1 \end{pmatrix} = \begin{pmatrix} -2+2i & 2 \\ -5-7i & 5-i \end{pmatrix}.$$

The next topic that we want to discuss here is powers of z.

The n- *th power of complex number* z we construct as:

$$z^n = \underbrace{z\cdot z\cdots z}_{n}.$$

Let's just take a look at what happens when we start looking at various powers of i:

$$i^0 = 1, \quad i^1 = i, \quad i^2 = -1, \quad i^3 = -i,$$

$$i^4 = 1, \quad i^5 = i, \quad i^6 = -1, \quad i^7 = -i,$$

$$i^8 = 1, \quad i^9 = i, \quad i^{10} = -1, \quad i^{11} = -i,$$

…………………………………… .

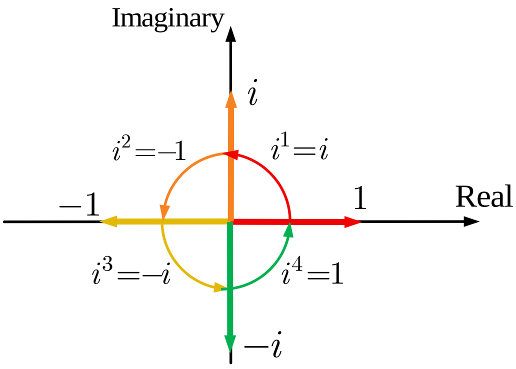

Fig. 3.3

It is evident that powers of i form cycle of length 4 (Fig. 3.3), i.e.

$$i^n = \begin{cases} 1, & n = 4k, \\ i, & n = 4k+1, \\ -1, & n = 4k+2, \\ -i, & n = 4k+3, \quad k \in \mathbb{N}. \end{cases}$$

Let’s simplify

$$i^{123} = i^{120+3} = i^{4\cdot 30+3} = i^3 = -i\,.$$

Using powers of i, we obtain:

$$z^2 = (x+iy)^2 = (x^2 - y^2) + i\cdot(2xy)$$
$$z^3 = (x+iy)^3 = (x^3 - 3xy^2) + i\cdot(3x^2y - y^3)$$

Now we’ll define division of complex numbers as the inverse of multiplication.

We begin with some preliminary ideas.

The conjugate of a complex number $z = x + iy$ is a complex number $x - iy$. In other words, it is the original complex number with the imaginary part negated. The conjugate of a complex number $z = x + iy$ is denoted by the symbol $\overline{z}$ (read “z bar” or “z conjugate”):

$$\overline{z} = x - iy\,.$$

Thus, the complex number $\overline{z} = 3 + i$ is conjugate of the complex number $z = 3 - i$, and for $z = 2$ the complex conjugate is $\overline{z} = 2$, i.e. the conjugate of a real number is just itself with no changes.

Complex conjugation has the following properties:

$$\overline{\overline{z}} = z;$$

$$\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2;$$

$$\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2;$$

$$\overline{z} = z \Leftrightarrow z = x \in \mathbb{R};$$

$$z + \overline{z} = 2x = 2\,\mathrm{Re}\,z \Rightarrow x = \frac{z + \overline{z}}{2};$$

$$z - \overline{z} = 2iy = 2i\,\mathrm{Im}\,z \Rightarrow y = \frac{z - \overline{z}}{2i}.$$

When we *multiply a non-zero complex number by its conjugate* we get *a positive real number* given by

$$z\overline{z} = (x + iy)(x - iy) = (x^2 - i^2 y^2) = x^2 + y^2.$$

! **N.B.** $$z\overline{z} = x^2 + y^2.$$

We now turn to the division of complex numbers.

For a nonzero complex number $z = x + iy \neq 0$ there exists the reciprocal of complex number. Indeed, multiplying the equality

$$z^{-1} z = 1$$

by $\overline{z}$, we get

$$z^{-1} z\overline{z} = \overline{z} \Rightarrow$$

$$z^{-1} = \frac{1}{z\bar{z}}\,\bar{z} = \frac{x}{x^2 + y^2} - i\,\frac{y}{x^2 + y^2}\,.$$

The quotient of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ $(z_2 \neq 0)$ is called the complex number

$$\frac{z_1}{z_2} = z_1 \cdot z_2^{-1} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\,.$$

Thus, the division of complex numbers is achieved by multiplying both numerator and denominator by the complex conjugate of the denominator:

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2}\,.$$

Example

Let's find $\dfrac{z_1}{z_2}$,

if

$$z_1 = 2 + i\,,\; z_2 = 7 - 3i\,.$$

❍ We have

$$\frac{z_1}{z_2} = \frac{2+i}{7-3i} = \frac{(2+i)\;\; 7+3i}{(7-3i)(7+3i)} =$$

$$= \frac{11+13i}{49+9} = \frac{11}{58} + \frac{13}{58}\,i\,.$$

●

3.1.3. Geometric Representation of Complex Numbers (Vector Model)

Jean-Robert Argand

For the first time, the geometric representation of operations with Complex Numbers was introduced by the Norwegian-Danish surveyor *Caspar Wessel* in 1799 and independently of it by the French mathematician *Jean-Robert Argand* in 1806.

Caspar Wessel

A complex number $z = x + iy$, as we know, is the ordered pair of real numbers (x, y). So it may be viewed as either *a point* $M(x; y)$ in the standard Cartesian Coordinate System, using the correspondence

$$x + iy \leftrightarrow (x, y),$$

or as *a radius-vector* $\overrightarrow{OM}(x; y)$ (Fig. 3.4).

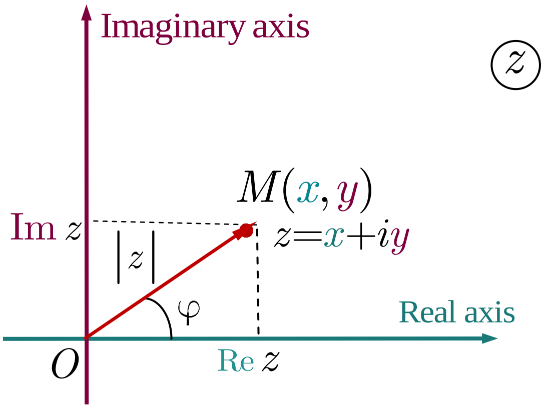

Fig. 3.4

In this interpretation we call the x-axis *the Real axis* and the y-axis we call *the Imaginary axis*. The representation is known as *the Argand diagram*, or *the Complex plane* $\mathbb{C}$, or *the plane* z.

N.B. *The imaginary unit* i *can be interpreted as the point* $(0,1)$, *thus breaking away from it the mystical color.*

The complex conjugate $\overline{z}$ *is obtained by reflecting* z *about the real axis* (Fig. 3.5).

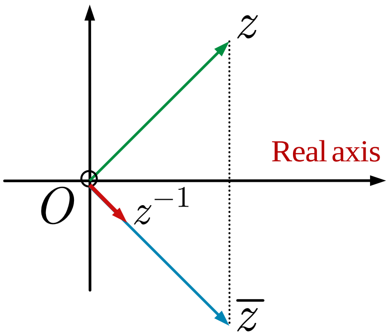

Fig. 3.5

Complex numbers are added and subtracted in the same way as vectors (Fig. 3.6):

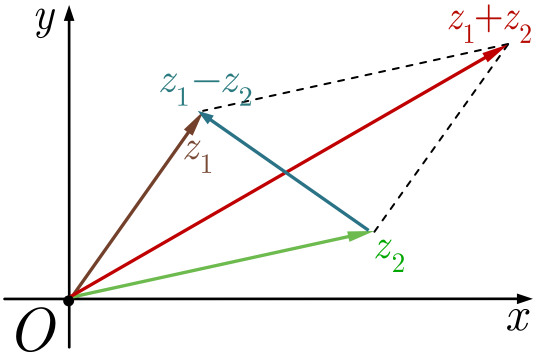

Fig. 3.6

Let's consider multiplication of complex numbers geometrically.

Multiplication of a complex number (a radius-vector on a plane) *by real number* (*regular multiplication*) scales up the complex number (makes a radius-vector larger or smaller).

For example, multiplication $1+3i$ by 2, i.e.

$$(1+3i)\cdot 2 = 2+6i$$

is represented geometrically on Fig. 3.7.

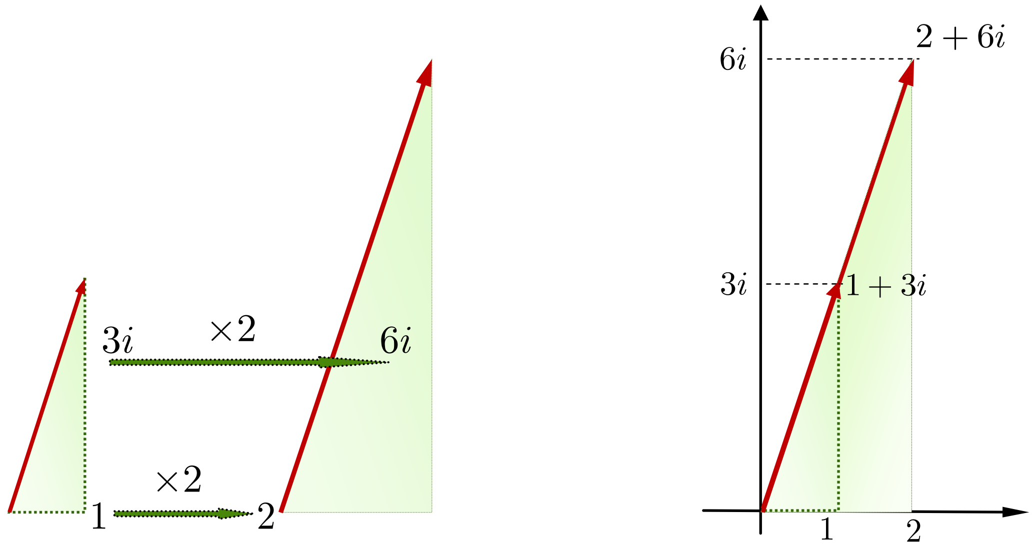

Fig. 3.7

Multiplication of a complex number by imaginary number i (*imaginary multiplication*) rotates a radius-vector by 90 degrees counter-clockwise. Visual representation of multiplication $1+3i$ by i

$$(1+3i)\cdot i = 1\cdot i + 3i\cdot i = -3+i$$

you may see on Fig. 3.8.

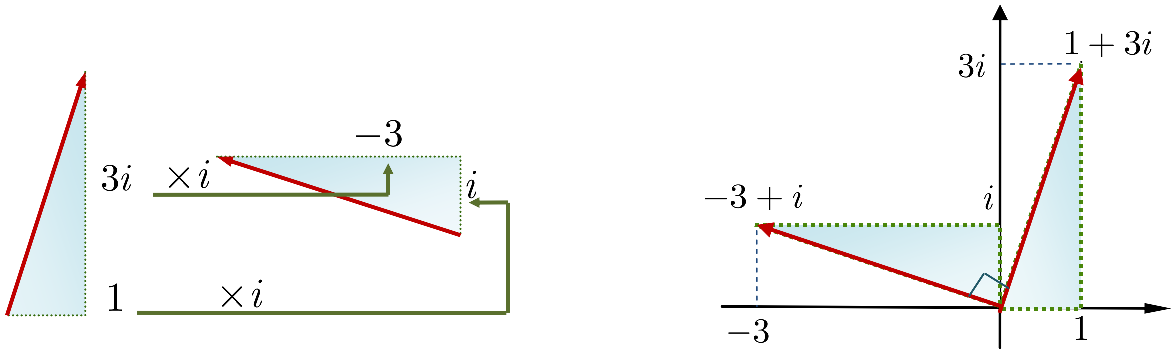

Fig. 3.8

And what if we combine the effects in a complex number? Multiplying the complex number $1+3i$ by $2+i$ means "double our number, then add in a perpendicular rotation" (Fig. 3.9):

$$(1+3i)\cdot(2+i) = (1+3i)\cdot 2 + (1+3i)\cdot i =$$

$$= (2+6i) + (-3+i) = -1+7i\,.$$

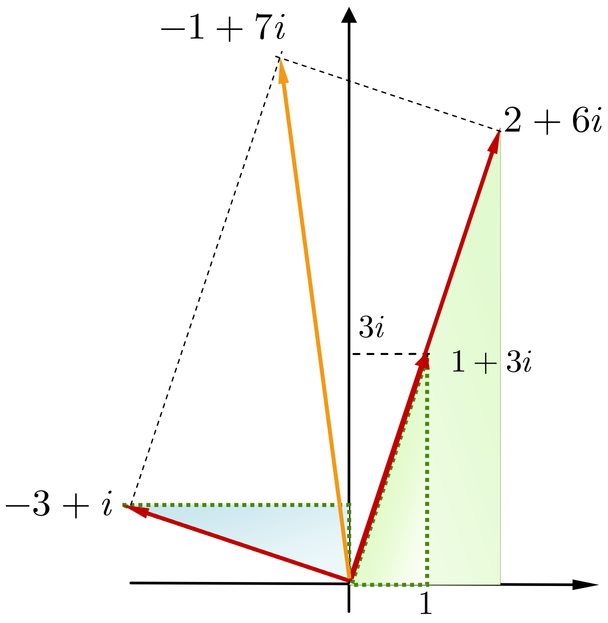

Fig. 3.9

We may give another geometrical representation of multiplication of complex numbers. We'll consider the product of the complex number $1+3i$ by the complex number $2+i$ in the following form:

$$(1+3i)\cdot(2+i) = 1\cdot 2 + 3i\cdot 2 + 1\cdot i + 3i\cdot i\,.$$

Multiplication by terms geometrically is represented on Fig. 3.10.

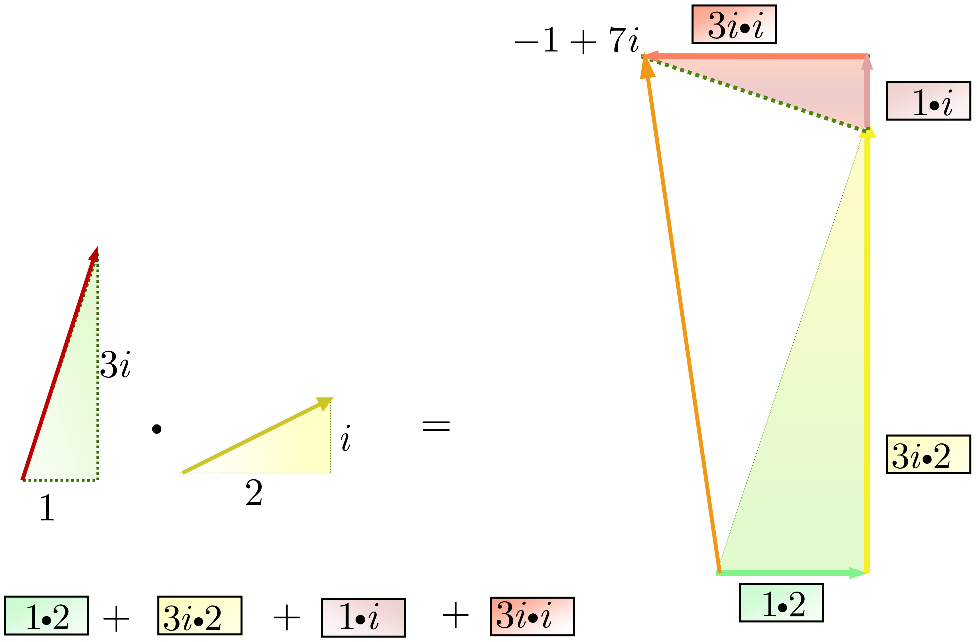

Fig. 3.10

The geometrical representation of complex numbers can be very useful when complex number methods are used to investigate properties of triangles and circles.

The length of the vector $\overrightarrow{OM}$ (see Fig. 3.4) is called *the modulus of a complex number* and is denoted by $\left|z\right|$:

$$\left|z\right| = \sqrt{x^2 + y^2}.$$

The concept of the modulus of a complex number is consistent with the concept of the modulus of a real number:

$$\left|x + 0i\right| = \sqrt{x^2 + 0^2} = \left|x\right|.$$

For modulus of sum of two complex numbers we obtain such inequality (see Fig. 3.6):

$$\left|z_1 + z_2\right| \le \left|z_1\right| + \left|z_2\right|.$$

From the last property of the conjugate complex number we obtain a useful formula:

$$z\overline{z} = \left|z\right|^2 \Rightarrow \frac{1}{z} = \frac{\overline{z}}{\left|z\right|^2}. \tag{3.1}$$

This means, the modulus of $\frac{1}{z}$ is the reciprocal of the modulus of z. For example, if $\left|z\right| = 2$, as in the Fig. 3.3, then $\left|\frac{1}{z}\right| = \frac{1}{2}$.

Corollary

The modulus of product of two complex numbers is equal to the the product of their modules:

$$\left|z_1 z_2\right| = \left|z_1\right|\left|z_2\right|.$$

► According to (3.1) we have:

$$\left|z_1 z_2\right|^2 = (z_1 z_2)\overline{(z_1 z_2)} = z_1 z_2 \overline{z}_1 \overline{z}_2 = z_1 \overline{z}_1 z_2 \overline{z}_2 = \left|z_1\right|^2 \left|z_2\right|^2,$$

hence,

$$\left|z_1 z_2\right|^2 = \left|z_1\right|^2 \left|z_2\right|^2 \Rightarrow \left|z_1 z_2\right| = \left|z_1\right|\left|z_2\right|.$$

If

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2,$$

then

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1),$$

and

$$|z_1|^2 = x_1^2 + y_1^2, \qquad |z_2|^2 = x_2^2 + y_2^2,$$

$$|z_1 z_2|^2 = (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2.$$

Hence, we obtain the interesting identity, which is called *the Euler's Identity*:

$$(x_1^2 + y_1^2)(x_2^2 + y_2^2) = (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2$$

(*the product of the sum of two squares by the sum of two squares is again the sum of two squares*). ◀

The angle φ between vector $\overrightarrow{OM}$ and the positive real axis Ox (see Fig. 3.4) is called *the argument of a complex number* and is denoted by $\mathrm{Arg}z$. The particular argument of complex number $z \neq 0$ lying in the range $-\pi < \varphi \leq \pi$ is called *the principal value of the argument* (*the principal argument*) of z and is denoted by $\arg z$. The argument of the complex number $z \neq 0$ is measured up to $2\pi k, k \in \mathbb{Z}$:

$$\mathrm{Arg}\, z = \arg z + 2\pi k, k \in \mathbb{Z}.$$

The principal value of the argument $\arg z = \varphi$ we can find solving the system:

$$\begin{cases} \cos\varphi = \dfrac{x}{\sqrt{x^2 + y^2}}, \\ \sin\varphi = \dfrac{y}{\sqrt{x^2 + y^2}}. \end{cases}$$

Let’s show some important examples.

1. Locus $|z-z_0|=r$ is the circle with centre in the point $z_0=x_0+iy_0$ and with radius r: $(x-x_0)^2+(y-y_0)^2=r^2$ (Fig. 3.11).

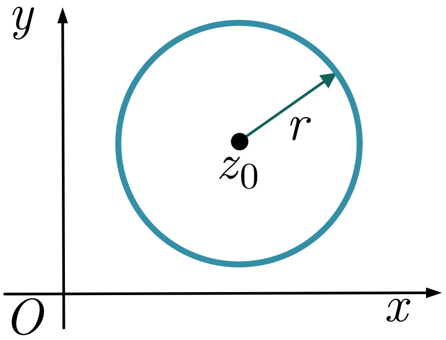

Fig. 3.11

2. An inequality $|z-z_0|< r$ represents the interior of the circle with the centre in the point z_0 and with radius r (Fig. 3.12).

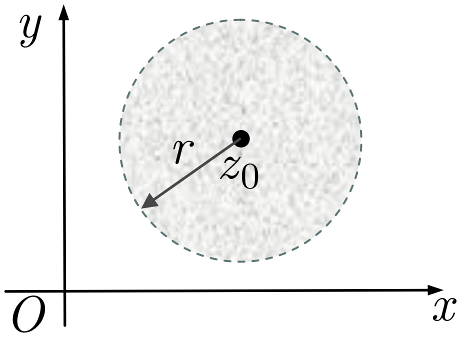

Fig. 3.12

3. An inequality $|z-z_0|>r$ represents the exterior of the circle with the centre in the point z_0 and with radius r (Fig. 3.13).

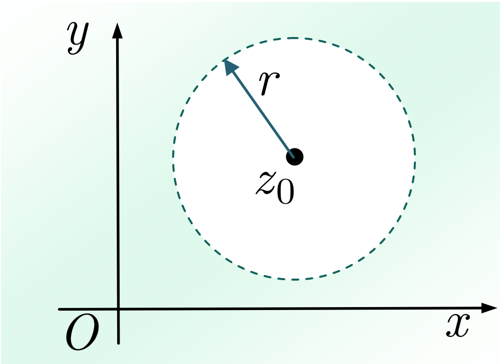

Fig. 3.13

4. An equation $\arg z = \alpha$ represents the ray from the origin at an angle α (Fig. 3.14).

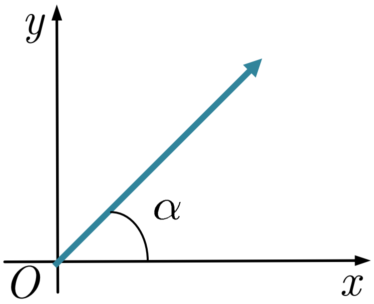

Fig. 3.14

5. An equation $\operatorname{Re} z = a$ represents the vertical line $x = a$ (Fig. 3.15).

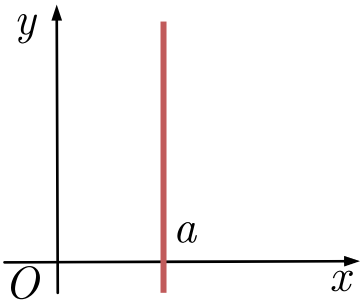

Fig. 3.15

6. An equation $\operatorname{Im} z = b$ represents the horizontal line $y = b$ (Fig. 3.16).

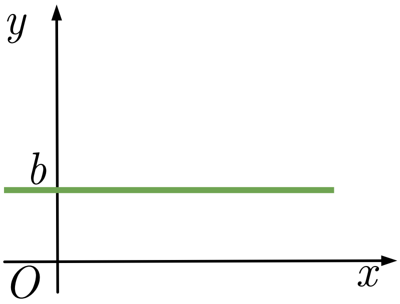

Fig. 3.16

Example

Let's try to depict on the Complex Plane a set of points that satisfy the next condition:

$$-2 < \operatorname{Im} z < 1, \quad 0 < \operatorname{Re} z < 3.$$

❍ As we know,

$$\operatorname{Im} z = y \text{ and } \operatorname{Re} z = x\,.$$

So, in our case we have (Fig. 3.17):

$$-2 < y < 1, \quad 0 < x < 3\,. \qquad \bullet$$

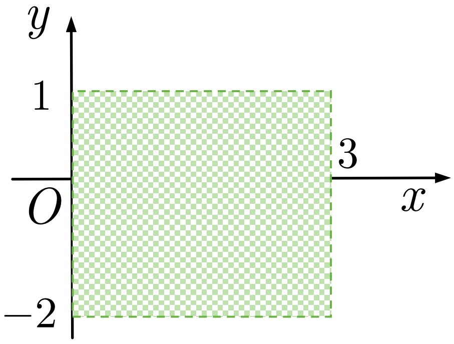

Fig. 3.17