We are familiar with complex numbers in the form $z = x + iy$, however there are some alternate forms that are useful at times. Now we'll look at both of those as well as a couple of nice facts that arise from them.

3.1.4. Matrix Model of Complex Numbers

We'll now use matrices to give representation of complex numbers.

Let us consider two Cartesian complex numbers:

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2.$$

By vector model given complex numbers we may write as vectors in $\mathbb{R}^2$:

$$z_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \text{and} \quad z_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

To discover a matrix form of a complex number let's first review complex multiplication.

As we know

$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

or in the vector form

$$z_1 \cdot z_2 = \begin{pmatrix} x_1x_2 - y_1y_2 \\ x_1y_2 + y_1x_2 \end{pmatrix}.$$

The latter expression is best represented as

$$\begin{pmatrix} x_1x_2 - y_1y_2 \\ y_1x_2 + x_1y_2 \end{pmatrix},$$

and it can be recognized as a matrix – vector product

$$\begin{pmatrix} x_1x_2 - y_1y_2 \\ y_1x_2 + x_1y_2 \end{pmatrix} = \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Here the first matrix is considered as coefficient matrix, the second matrix is the vector form of the complex number z_2.

Thus the product of the complex numbers z_1 and z_2 we can write as the pair of two matrices

$$z_1 \cdot z_2 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

We may assume that the coefficient matrix is *the matrix form* of the complex number z_1, i.e.

$$z_1 = x_1 + y_1 \cdot i = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \mapsto \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix}.$$

This leads us to possibility of modeling complex numbers by special square matrices of the second order:

$$z = x + iy = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

If

$$x = 1, \quad y = 0,$$

we get the first very important complex number $z = 1$ and its matrix representation

$$1 + 0 \cdot i = 1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I}.$$

We obtain the special matrix — well-known *Identity matrix*.

To get the second important complex number $z = i$ we put

$$x = 0, \; y = 1.$$

And we have

$$0 + 1 \cdot i = i \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus we get the matrix representation of complex numbers $z = 1$ and $z = i$:

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad i \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

We may verify that the main equality for i

$$i^2 = -1$$

holds in this case. Indeed,

$$i^2 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\mathrm{I} \mapsto -1.$$

So any complex number $z = x + iy$ we may represent in the following matrix form:

$$z = x \cdot 1 + y \cdot i = x \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Matrix representation of the complex number means that with usual addition and multiplication operations for matrices complex number satisfies all the complex number axioms and properties.

Let us show that multiplication of complex numbers in matrix form is commutative, i.e.

$$z_1 z_2 = z_2 z_1,$$

where

$$z_1 = \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \text{ and } z_2 = \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix}.$$

Indeed,

$$z_1 z_2 = \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix}\begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix} = \begin{pmatrix} x_1x_2 - y_1y_2 & -x_1y_2 - y_1x_2 \\ y_1x_2 + x_1y_2 & -y_1y_2 + x_1x_2 \end{pmatrix},$$

$$z_2 z_1 = \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix}\begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} = \begin{pmatrix} x_2x_1 - y_2y_1 & -x_2y_1 - y_2x_1 \\ y_2x_1 + x_2y_1 & -y_2y_1 + x_2x_1 \end{pmatrix}.$$

The matrices are equal.

To check the other properties we leave as exercises.

One essential feature of complex numbers is *a complex conjugate*:

$$\overline{z} = x - yi\,.$$

And what is the matrix representation of this complex conjugate? We have

$$z = x + yi \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix},$$

then we should get

$$\overline{z} = x - yi \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

This relation is called, as we know, *the transpose matrix*:

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

That's why, when we find the complex conjugate in matrix form, we simply transpose the matrix which represents the given complex number:

$$\overline{z} \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{\mathrm{T}}.$$

Another important feature of a complex number is its *modulus*. As we know, the numerical characteristic of a matrix is a determinant.

Let's find *the determinant* of the matrix $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$:

$$\begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 - (-y^2) = x^2 + y^2\,.$$

You are wonder, but we have got the square of the modulus of our complex number. Thus,

$$|z| = \sqrt{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}} = \sqrt{x^2 + y^2}\,.$$

Let us find the inverse of the matrix – complex number.

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1} = \frac{1}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \frac{1}{x^2+y^2} \begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{\mathrm{T}}.$$

We recognize that this *inverse matrix* is the usual *reciprocal* of a complex number:

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{1}{x^2+y^2} \begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1}.$$

Corollary

1. *Adding or multiplying two complex numbers is the same as adding or multiplying their matrix representations.*

2. *Dividing by a complex number is the same as multiplying by the inverse of its matrix representation.*

3. *Conjugating a complex number is the same as transposing its matrix representation.*

4. *The modulus of a complex number is the same as the determinant of its matrix representation.*

Remark

This matrix representation of complex numbers is not the only one.

3.1.5. Polar Form of Complex Number

Let $z = x + iy$ be a Cartesian Complex Number. According to geometric representation of complex number z we have the point $M(x, y)$ on the Complex plane (Fig. 3.18). This point can be represented by the polar coordinates $(\rho; \varphi)$:

$$x = \rho \cos \varphi\,, \; y = \rho \sin \varphi \;.$$

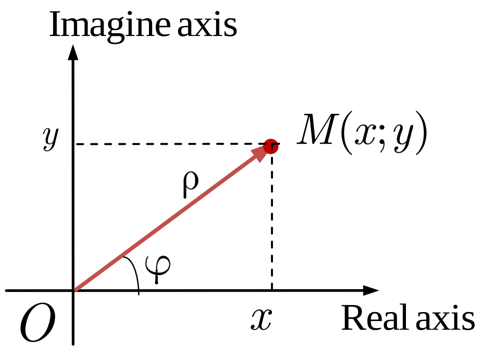


Fig. 3.18

Hence, a complex number $z = x + iy$ can be written in the form:

$$z = x + iy = \rho \cos \varphi + i\rho \sin \varphi = \rho(\cos \varphi + i \sin \varphi), \rho \geq 0\,.$$

The expression

$$z = \rho(\cos \varphi + i \sin \varphi)$$

is called *the Polar Form of a complex number* z.

We can also represent a complex number in matrix polar form

$$z = \rho \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}$$

or as a product of two matrices

$$z = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} = \mathrm{H}(\rho) \cdot \mathrm{R}(\varphi)\,.$$

Here the matrix $\mathrm{H}(\rho)$ corresponds to a linear transformation, which describes *homothety* with the center at the origin and the similarity coefficient $\rho > 0$. The matrix $\mathrm{R}(\varphi)$ corresponds to a linear transformation, that describes the *rotation* of the plane around the origin by the angle φ counterclockwise.

The matrix $R(\varphi)$ has very interesting property. Let's find the product of the matrix $R(\varphi)$ and the transpose matrix $R^T(\varphi)=\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}$:

$$R(\varphi)\cdot R^T(\varphi)=\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=I\,.$$

We obtain the Identity matrix. It means that the transpose matrix equals to the inverse matrix:

$$R^T(\varphi)=R^{-1}(\varphi)\,.$$

N.B. *Such matrix is called the Orthogonal matrix.*
The rotation matrix is the orthogonal matrix.

Thus, the complex number defines some rotational homothety relative to the origin, characterized by the angle of rotation and the coefficient of rotation.

The quantity

$$\rho=\sqrt{x^2+y^2}\ge 0\,,$$

as we know, is called *the modulus of a complex number* z ($|z|$).

The unique value of the angle $\varphi\in(-\pi,\pi]$ is *the principal value of the argument* $(\arg z)$ (then we will call it simply as *the argument* of a complex number).

We note that $\dfrac{y}{x}=\mathrm{tg}\varphi$, if $x\ne 0$, so φ is determined by this equation up to a multiple of π. In fact

$$\arg z=\mathrm{arctg}\frac{y}{x}+k\pi\,,$$

where $k=0$ if $x>0$; $k=1$ if $x<0$, $y>0$; $k=-1$ if $x<0$, $y<0$.

One has to apply the following rules in order to determine $\arg z \in (-\pi, \pi]$ as:

$$\varphi = \begin{cases} \operatorname{arctg} \dfrac{y}{x}, & x > 0, \\ \dfrac{\pi}{2}, & x = 0, y > 0, \\ \operatorname{arctg} \dfrac{y}{x} + \pi, & x < 0, y > 0, \\ \operatorname{arctg} \dfrac{y}{x} - \pi, & x < 0, y < 0, \\ -\dfrac{\pi}{2}, & x = 0, y < 0. \end{cases}$$

Example

Let's find the polar form of the complex number

$$z = -1 - i\sqrt{3}\,.$$

❍ We have

$$z = -1 - i\sqrt{3}: \qquad x = \operatorname{Re} z = -1, \quad y = \operatorname{Im} z = -\sqrt{3}\,.$$

Let's first find the modulus ρ:

$$\rho = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{4} = 2\,.$$

Now let's find the argument of z. Since

$$x < 0, y < 0 \Rightarrow z \in \text{III quadrant}\,.$$

Hence, the argument of the complex number z we'll find as

$$\varphi = \operatorname{arctg}\left(\frac{-\sqrt{3}}{-1}\right) - \pi = \frac{\pi}{3} - \pi = -\frac{2\pi}{3}.$$

Now, let's actually do what we were originally asked to do. Here is the polar form of the complex number $z = -1 - i\sqrt{3}$:

$$z = 2(\cos(-\frac{2\pi}{3}) + i\sin(-\frac{2\pi}{3}))\,. \qquad \bullet$$

Operations with Complex Numbers in Polar Form

For two complex numbers in polar form

$$z_1 = \rho_1(\cos\varphi_1 + i\sin\varphi_1),$$

$$z_2 = \rho_2(\cos\varphi_2 + i\sin\varphi_2)$$

conditions of equality, the formula of multiplication and division look like this:

$$z_1 = z_2 \Leftrightarrow \begin{cases} \rho_1 = \rho_2, \\ \varphi_1 = \varphi_2 + 2\pi k, k \in \mathbb{Z}, \end{cases}$$

$$z_1 z_2 = \rho_1\rho_2(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)),$$

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2}(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)).$$

► Let us prove, for example, the second equality:

$$z_1 z_2 = \rho_1(\cos\varphi_1 + i\sin\varphi_1)\rho_2(\cos\varphi_2 + i\sin\varphi_2) =$$

$$= \rho_1\rho_2[(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2) + i(\sin\varphi_1\cos\varphi_2 + \sin\varphi_2\cos\varphi_1)] =$$

$$= \rho_1\rho_2[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)].$$

Geometrically this result means: in order to obtain the product of a complex number

$$z_1 = \rho_1(\cos\varphi_1 + i\sin\varphi_1)$$

by a complex number

$$z_2 = \rho_2(\cos\varphi_2 + i\sin\varphi_2),$$

it is necessary to rotate the ray Oz_1 at an angle φ_2 counterclockwise, mark the point that is at a distance $\rho_1\rho_2$ from the origin on the received ray (Fig. 3.19). ◄

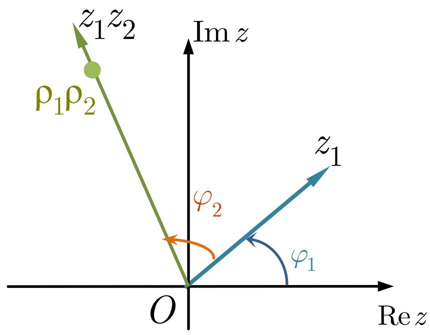


Fig. 3.19

Remark

Multiplying complex numbers we multiply their modules and add their arguments.

When we divide complex numbers in polar form their modules are divided, and their arguments are subtracted.

Example

Let us express the complex number

$$(\sqrt{3}-i)(3+3i)$$

in polar form.

- Let us first express the complex number

$$z_1 = \sqrt{3} - i$$

in polar form:

$$\operatorname{Re} z_1 = \sqrt{3} > 0\,,\;\; \operatorname{Im} z_1 = -1 < 0 \Rightarrow z \in \text{IV quadrant:}$$

$$\rho_1 = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2\,,\;\; \varphi_1 = \operatorname{arctg}(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}:$$

$$z_1 = 2\left(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right).$$

And now we let's find the Polar form of the complex number $z_2 = 3 + 3i$:

$$\operatorname{Re} z_2 = 3 > 0, \operatorname{Im} z_2 = 3 > 0 \Rightarrow z \in \text{I quadrant:}$$

$$\rho_2 = \sqrt{3^2 + 3^2} = 3\sqrt{2}\,, \quad \varphi_2 = \operatorname{arctg}\frac{3}{3} = \frac{\pi}{4}\,,$$

$$z_2 = 3\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})\,.$$

Then

$$z_1 z_2 = 2\left(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right) \cdot 3\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) =$$

$$= 6\sqrt{2}\left(\cos\left(-\frac{\pi}{6} + \frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{6} + \frac{\pi}{4}\right)\right) = 6\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right). \quad \bullet$$

And now we're going to take a look at a really nice way of quickly computing integer powers and roots of complex numbers.

Power of Complex Number

According to the definition of the natural power n of a complex number

$$z = \rho(\cos\varphi + i\sin\varphi)$$

we obtain:

$$z^n = \left(\rho(\cos\varphi + i\sin\varphi)\right)^n = \rho^n(\cos n\varphi + i\sin n\varphi) \Rightarrow$$

$$\Rightarrow \left|z^n\right| = \rho^n,\ \arg(z^n) = n\arg z\,.$$

The formula

$$z^n = \rho^n(\cos n\varphi + i\sin n\varphi)$$

is called *De Moivre's formula*.

Abraham De Moivre

► It's evident, that De Moivre's formula is true for $n = 1$:

$$(\rho(\cos\varphi + i\sin\varphi))^1 = \rho(\cos\varphi + i\sin\varphi).$$

Suppose, that this formula is valid, when $n = k$, i.e.

$$(\rho(\cos\varphi + i\sin\varphi))^k = \rho^k(\cos k\varphi + i\sin k\varphi).$$

Then prove, that De Moivre's formula is true, when $n = k+1$:

$$(\rho(\cos\varphi + i\sin\varphi))^{k+1} = \rho(\cos\varphi + i\sin\varphi)(\rho(\cos\varphi + i\sin\varphi))^k =$$

$$= \rho(\cos\varphi + i\sin\varphi)\rho^k(\cos k\varphi + i\sin k\varphi) =$$

$$= \rho^{k+1}(\cos k\varphi\cos\varphi - \sin\varphi\sin k\varphi + i(\sin\varphi\cos k\varphi + \cos\varphi\sin k\varphi)) =$$

$$= \rho^{k+1}(\cos(k+1)\varphi + i\sin(k+1)\varphi).$$

Hence De Moivre's formula is valid for any natural n. ◄

Remark

N.B. *When the complex number is raised to a positive integer power n the modulus is raised to this power n, while the argument is multiplied by n.*

Example

Let us calculate

$$\left(\frac{3+3i}{\sqrt{3}-i}\right)^6.$$

❍ Let's write the complex numbers $z_1 = 3+3i$ and $z_2 = \sqrt{3}-i$ in a polar form (see the results of the above example):

$$z_1 = 3\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right), \qquad z_2 = 2(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6}))\ .$$

Then we have

$$\frac{z_1}{z_2} = \frac{3\sqrt{2}\,(\cos\frac{\pi}{4} + \sin\frac{\pi}{4})}{2\,(\cos(-\frac{\pi}{6}) + \sin(-\frac{\pi}{6}))} = \frac{3}{\sqrt{2}}(\cos(\frac{\pi}{4} + \frac{\pi}{6}) + \sin(\frac{\pi}{4} + \frac{\pi}{6})) =$$

$$= \frac{3}{\sqrt{2}}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right).$$

Thus

$$\left(\frac{3+3i}{\sqrt{3}-i}\right)^6 = \left(\frac{3}{\sqrt{2}}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)\right)^6 =$$

$$= \frac{3^6}{8}\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = \frac{3^6}{8}\,i. \qquad \bullet$$

Remark

De Moivre's formula is also valid for all integers n ($n \in \mathbb{Z}$):

$$z^0 = 1;$$

$z \neq 0$:

$$z^{-1} = \frac{1}{z} = \frac{1}{\rho(\cos\varphi + i\sin\varphi)} =$$

$$= \frac{1}{\rho} \cdot \frac{\cos\varphi - i\sin\varphi}{\cos^2\varphi + \sin^2\varphi} = \rho^{-1}(\cos(-\varphi) + i\sin(-\varphi)),$$

$n = -m,\ m \in \mathbb{N}$:

$$z^n = z^{-m} = (z^{-1})^m = \rho^{-m}(\cos m(-\varphi) + i\sin m(-\varphi)) =$$

$$= \rho^{-m}(\cos(-m\varphi) + i\sin(-m\varphi)) = \rho^n(\cos n\varphi + i\sin n\varphi).$$

If we take $\rho = 1$ in De Moivre's formula, we obtain

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi\,.$$

If we expand the left side of De Moivre's formula above by the Binomial formula and reduce it to the Cartesian form, we obtain formulas for $\cos n\varphi$ and $\sin n\varphi$ as polynomials of degree n in $\cos\varphi$ and $\sin\varphi$.

For example, if $n = 3$, we have

$$(\cos\varphi + i\sin\varphi)^3 = \cos 3\varphi + i\sin 3\varphi\,.$$

The left side of this equation expands to

$$\cos^3\varphi + 3i\cos^2\varphi\sin\varphi - 3\cos\varphi\sin^2\varphi - i\sin^3\varphi\,.$$

The real part of this must equal $\cos 3\varphi$ and the imaginary part must equal $\sin 3\varphi$. Therefore,

$$\cos 3\varphi = \cos^3\varphi - 3\cos\varphi\sin^2\varphi =$$
$$= \cos\varphi(\cos^2\varphi - 3\sin^2\varphi) = \cos\varphi(4\cos^2\varphi - 3)\,;$$
$$\sin 3\varphi = 3\cos^2\varphi\sin\varphi - \sin^3\varphi =$$
$$= \sin\varphi(3\cos^2\varphi - \sin^2\varphi) = \sin\varphi(3 - 4\sin^2\varphi)\,.$$

Root of Complex Number

The n*-th root of a complex number* z is called the complex number w

$$w = \sqrt[n]{z}\,,$$

whose n-th power is equal to the radicand

$$w^n = z\,. \tag{3.2}$$

If $z \neq 0$ any root $\sqrt[n]{z}$ has n different values.
Let

$$z = \rho(\cos\varphi + i\sin\varphi)\,, \qquad w = r(\cos\theta + i\sin\theta)$$

Consequently, w satisfies (3.2) then we get:

$$r^n(\cos n\theta + i\sin n\theta) = \rho(\cos\varphi + i\sin\varphi)\,. \tag{3.3}$$

We satisfy equation (3.3) if

$$r^n = \rho \quad \text{and} \quad n\theta = \varphi + 2\pi k,\, k \in \mathbb{Z},$$

because the modules of equal complex numbers must be equal, while the addition of any integer multiple of 2π to the argument is also a solution.
Thus,

$$r = \sqrt[n]{\rho}\,,$$

where r is the uniquely determined real positive number, and

$$\theta = \frac{\varphi + 2\pi k}{n},\, k \in \mathbb{Z}.$$

There might appear to be infinitely many different answers ω_k corresponding to the infinitely many possible values of k.
Giving k the values $0, 1, 2, \ldots, n-1$, we get n different complex numbers ω_k:

$$k = 0: \;\; \theta_0 = \frac{\varphi}{n};$$

$$k=1:\quad \theta_1=\frac{\varphi+2\pi}{n}=\frac{\varphi}{n}+\frac{1}{n}\cdot 2\pi\,;$$

$$k=2:\quad \theta_2=\frac{\varphi+4\pi}{n}=\frac{\varphi}{n}+\frac{2}{n}\cdot 2\pi;$$

$$k=n-1:\quad \theta_{n-1}=\frac{\varphi+(n-1)2\pi}{n}=\frac{\varphi}{n}+\frac{n-1}{n}\cdot 2\pi\,.$$

When

$$k=n:\quad \theta_n=\frac{\varphi+2\pi n}{n}=\frac{\varphi}{n}+2\pi\,,$$

we have

$$\cos\theta_n=\cos(\frac{\varphi}{n}+2\pi)=\cos\frac{\varphi}{n}=\cos\theta_0$$

and

$$\sin\theta_n=\sin(\frac{\varphi}{n}+2\pi)=\sin\frac{\varphi}{n}=\sin\theta_0\,.$$

Therefore, there are exactly n different roots

$$\sqrt[n]{z}=\sqrt[n]{\rho}\left(\cos\left(\frac{\varphi+2\pi k}{n}\right)+i\sin\left(\frac{\varphi+2\pi k}{n}\right)\right),\ k=\overline{0,n-1}\,. \qquad (3.4)$$

Remark

The modules of all roots of the complex number z are the same, but the arguments are different, they are $\frac{2\pi k}{n}$ apart.

The roots on the complex plane lye in the vertices of the regular n - gon, that is inscribed in the circle with radius $\sqrt[n]{\rho}$ and with centre in the point $(0;0)$.

Example

Let's find all the values of $\sqrt[4]{-16}$.

❍ Since

$$-16 = 16(\cos\pi + i\sin\pi)\,,$$

we obtain

$$\sqrt[4]{-16} = \sqrt[4]{16}(\cos\frac{\pi + 2\pi k}{4} + i\sin\frac{\pi + 2\pi k}{4})\,,\ \ k = 0,1,2,3\,;$$

or

$$\sqrt[4]{-16} = 2(\cos\frac{\pi + 2\pi k}{4} + i\sin\frac{\pi + 2\pi k}{4})\,,\ \ k = 0,1,2,3\,,$$

and we get

$$\omega_0 = 2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = \sqrt{2} + i\sqrt{2}\,;$$

$$\omega_1 = 2(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}) = -\sqrt{2} + i\sqrt{2}\,;$$

$$\omega_2 = 2(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}) = 2(\cos(-\frac{3\pi}{4}) + i\sin(-\frac{3\pi}{4})) = -\sqrt{2} - i\sqrt{2}\,;$$

$$\omega_3 = 2(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}) = 2(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})) = \sqrt{2} - i\sqrt{2}\,.$$

All roots $\sqrt[4]{-16}$ are located in the vertices of the square (Fig. 3.20). ●

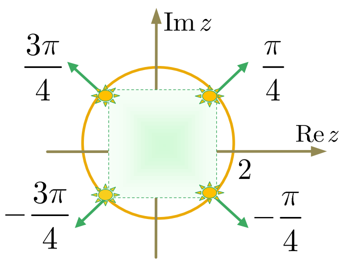
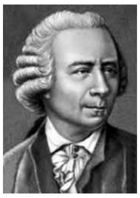


Fig. 3.20

3.1.6. Exponential Form of Complex Number

Now that we've discussed the polar form of a complex number we can introduce the next alternate form of a complex number.

First, we'll need *Euler's formula*, one of the famous achievements of Leonard Euler (also known as *the most beautiful equation in mathematics*):



Leonard Euler

$$e^{i\varphi} = \cos\varphi + i\sin\varphi, \varphi \in \mathbb{R}. \qquad (3.5)$$

Let us check this formula.
If $\varphi = 0$ we get:

$$e^{i0} = \cos 0 + i\sin 0 \quad \Rightarrow \quad 1 = 1.$$

For $\varphi = \pi$ we obtain

$$e^{i\pi} = \cos\pi + i\sin\pi \quad \Rightarrow$$

N.B.

$$e^{i\pi} + 1 = 0.$$

❗ *This equality includes four world constants: of geometry* (π), *of algebra* (1), *of analyses* (e), *of the theory of complex numbers* (i).

Geometrically, Euler's formula says: $e^{i\varphi}$ lies on the unit circle in $\mathbb{C}$ (Fig. 3.21):

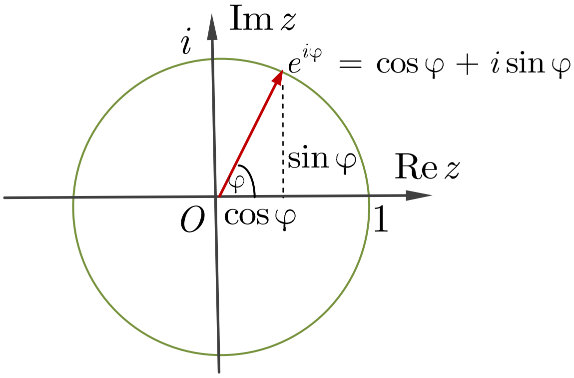


Fig. 3.21

In particular, if we multiply a given complex number z by $e^{i\varphi}$, which has unit length 1, the result:

$$e^{i\varphi} \cdot z$$

has the same length as z. *The complex number z is rotated by φ degrees.*

Remark

Euler's formula (3.5) *can be represented by the matrix form*

$$\exp(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

or

$$\exp\begin{pmatrix} 0 & -\varphi \\ \varphi & 0 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

The right side is a well-known 2-d rotation matrix.

With Euler's formula we can rewrite the polar form of a complex number

$$z = \rho(\cos\varphi + i\sin\varphi)$$

into its *exponential form* as follows:

$$z = \rho e^{i\varphi},$$

where

$$\rho = |z|, \varphi = \operatorname{Arg} z.$$

For complex numbers in exponential form the following identities are valid:
if

$$z_1 = \rho_1 e^{i\varphi_1}, \qquad z_2 = \rho_2 e^{i\varphi_2};$$

then

$$z_1 = z_2 \Leftrightarrow \rho_1 = \rho_2,\ \varphi_1 = \varphi_2 + 2\pi k;$$

$$\overline{z}_1 = \rho_1 e^{i(-\varphi_1)}; \qquad \overline{z}_2 = \rho_2 e^{i(-\varphi_2)};$$

$$z_1 \cdot z_2 = \rho_1 \rho_2 e^{i(\varphi_1 + \varphi_2)};$$

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2} e^{i(\varphi_1 - \varphi_2)};$$

$$z_1^n = \rho_1^n e^{in\varphi_1};$$

$$\sqrt[n]{z_1} = \sqrt[n]{\rho_1} \cdot e^{i\frac{\varphi_1 + 2\pi k}{n}}, \quad k = 0, 1, \ldots, n-1.$$

Remark

Let's add and then subtract two equalities

$$e^{i\varphi} = \cos\varphi + i\sin\varphi,$$

$$e^{-i\varphi} = \cos\varphi - i\sin\varphi.$$

We get

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2};$$

$$\sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}.$$