3.2. Quaternions

- *Definition of Quaternion*
- *Basic properties*
- *Additive Form of Quaternion*
- *Matrix Form of Quaternion*
- *Polar Form of Quaternion*
- *Euler's Identity*

The theory of quaternions was introduced in the nineteenth century, and it found many applications in classical mechanics, quantum mechanics, and the theory of relativity. Quaternions were also later used in aerospace applications and flight simulators. Graphics and game programmers discovered the true potential of quaternions and started using them as a powerful tool for describing rotations about an arbitrary axis. From computer graphics, the application domain of quaternions soon expanded into other fields such as visualization, fractals and virtual reality.

The discovery and the name of the quaternion himself owes to the Irish mathematician *William Rowan Hamilton*. He spent years trying to extend the complex numbers to a new algebraic structure with each element consisting of one real part and two distinct imaginary parts. This would be known as the theory of “triplets”. He hoped to climb out of the complex plane by adding an imaginary j . But Hamilton worked unsuccessfully at creating this algebra.


William Hamilton

On October 16, 1843, while walking with his wife, Lady Hamilton, along the Royal Canal in Dublin it occurred to Hamilton that his new algebra would require three rather than two imaginary parts.

In a letter to his son, W. Hamilton wrote: "It was the 16th day of October, which took place on Monday, the day of the meeting of the Council of the Royal Irish Academy, where I had to vote. I went there with your mother along the Royal Canal. She told me some specific phrases, but I almost did not perceive them, because something was happening in my mind. Unexpectedly, as if the electric contour of thought was locked up, glittering sparks were the fundamental relations for the symbols".

At the same time, Hamilton made a significant act of mathematical vandalism — he carved on the soft stone of the Brougham Bridge (now Broome Bridge) of the Roal Canal:

$$i^2 = j^2 = k^2 = ijk = -1.$$

This event is marked by a plaque at the exact location today.

“Here as he walked by on the 16-th of October 1843 Sir William Rowan Hamilto in a flash of genius discovered the fundamental formula for quaternion multiplication and cut it on a stone of this bridge”.

Hamilton’s discovery was so significant that every year on October 16th, the Mathematics Department of the National University of Ireland commemorates his discovery by walking to Broome Bridge.

3.2.1. Basic Concepts

Definition of Quaternion

To better understand quaternion, it is important to understand it as an algebraic structure.

As we indicated, quaternions are a four dimensional extension of complex numbers with three dimensions being imaginary and the other being real. Therefore, many of the properties of quaternion algebra can be deduced by logically extending the well known properties of complex algebra. Unlike complex numbers, the set of quaternions are not commutative algebra under multiplication.

Quaternions can be defined in several different, equivalent ways. It is helpful to know them all, since each form is useful.

Definition 3.2.

Quaternion is called the expression of the form

$$q = x_0 + x_1 i + x_2 j + x_3 k\,,$$

where the real numbers x_0, x_1, x_2, x_3 *are called quaternion coordinates;*
the imaginary units i, j, k *satisfy the following conditions* (*Hamilton's Rules*):

$$\begin{gathered} i^2 = j^2 = k^2 = -1\,, \\ i \cdot j = k\,,\ j \cdot k = i\,,\ k \cdot i = j\,, \\ i \cdot j = -j \cdot i\,, \quad j \cdot k = -k \cdot j\,, \quad k \cdot i = -i \cdot k\,, \\ i \cdot j \cdot k = -1\,. \end{gathered} \tag{3.6}$$

The real number x_0 is called *the real* component of a quaternion q and the sum $x_1 i + x_2 j + x_3 k$ is called *the imaginary* part of this quaternion.

Thus, quaternions have 4 dimensions (each quaternion consists of 4 scalar numbers, hence the name "quat-"): one real dimension and 3 imaginary dimensions. Each of these imaginary dimensions has a unit value of the square root of (-1), but they are different square roots of (-1), all mutually perpendicular to each other.

N.B. *The quaternions don't commute*:

$$q_1 \cdot q_2 \neq q_2 \cdot q_1\,.$$

A useful mnemonic for multiplication i, j, k you may see on Fig. 3. 22:

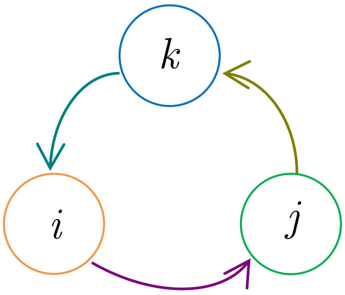

Fig. 3.22

You may also recognize i, j and k as *the standart orthonormal basis for* $\mathbb{R}^3$. And their product is the same as the cross product of unit vectors:

$$[\vec{i},\vec{j}] = \vec{k}\ ,\ [\vec{j},\vec{k}] = \vec{i}\ ,\ [\vec{k},\vec{i}] = \vec{j}\ .$$

Except, for the cross product:

$$[\vec{i},\vec{i}] = \vec{0}\ ,\ [\vec{j},\vec{j}] = \vec{0},\ [\vec{k},\vec{k}] = \vec{0}$$

while for quaternions, this is ❗ (-1).

In fact, we can think of a quaternion as having *a scalar part* x_0 and *a vector part* $\vec{x} = x_1 i + x_2 j + x_3 k$. So quaternion can be represented as follows:

$$q = x_0 + \vec{x} = \{x_0, \vec{x}\}\ .$$

If the vector part of a quaternion is zero ($\vec{x} = x_1 i + x_2 j + x_3 k = \vec{0}$), that is, if

$$q = \{x_0, \vec{0}\}\ ,$$

then a quaternion is identified with a real number

$$q = x_0$$

(*scalar quaternion*).

The quaternion

$$q = \{1, \vec{0}\} = \{1, 0, 0, 0\}$$

is called *the identical quaternion*.

When the real part of a quaternion is zero ($x_0 = 0$), then the quaternion

$$q = \{0,\ \vec{x}\}$$

is identified with a vector

$$q = \vec{x}$$

that has cartesian coordinates x_1, x_2, x_3 (*vector* (*right*) *quaternion*).

Due to quaternions are an extension of complex numbers, many properties of them are familiar.

Equality of quaternions

Two quaternions

$$q_1 = x_0 + x_1 i + x_2 j + x_3 k = \{x_0,\ \vec{x}\}$$

and

$$q_2 = y_0 + y_1 i + y_2 j + y_3 k = \{y_0,\ \vec{y}\}$$

are defined to be *equal*, if their real (scalar) parts are equal and their imaginary (vector) parts are equal too:

$$q_1 = q_2 \Leftrightarrow \begin{cases} x_0 = y_0, \\ \vec{x} = \vec{y} \end{cases} \Leftrightarrow \begin{cases} x_0 = y_0, \\ x_1 = y_1, \\ x_2 = y_2, \\ x_3 = y_3. \end{cases}$$

Addition and Subtraction of Quaternions

Addition and substraction of two quaternions act similar to complex numbers — component-wise:

$$q_1 \pm q_2 = (x_0 \pm y_0) + (x_1 \pm y_1)i + (x_2 \pm y_2)j + (x_3 \pm y_3)k\,,$$

or

$$q_1 \pm q_2 = (x_0 \pm y_0) + (\vec{x} \pm \vec{y})\,.$$

Additive Form of Quaternion

We may represent a quaternion in the following form

$$q = x_0 + \vec{x} = \{x_0, \vec{0}\} + \{0, \vec{x}\}\,,$$

which is called *the additive form* of a quaternion.

Quaternion Products

Multiplication of quaternions by a scalar is the same as for complex numbers:

$$\alpha q = \alpha x_0 + \alpha\vec{x} = \alpha x_0 + \alpha x_1 i + \alpha x_2 j + \alpha x_3 k\,,\ \alpha \in \mathbb{R}\ .$$

If q_1 and q_2 are scalar quaternions

$$q_1 = x_0\,,\ q_2 = y_0\,,$$

then their product is also a scalar quaternion

$$q_1 \cdot q_2 = x_0 y_0\,.$$

We can also express the product of a scalar quaternion

$$q_1 = x_0$$

and a vector quaternion

$$q_2 = \vec{y} = y_1 i + y_2 j + y_3 k$$

as

$$q_1 \cdot q_2 = x_0(y_1 i + y_2 j + y_3 k) = (x_0 y_1)i + (x_0 y_2)j + (x_0 y_3)k\,.$$

In this case we obtain the vector quaternion and the multiplication operation coincides with the multiplication of a vector $\vec{y}$ by a scalar x_0 in space.

Let us consider now the multiplication of two vector quaternions:

$$q_1 = \vec{x} = x_1 i + x_2 j + x_3 k \quad \text{and} \quad q_2 = \vec{y} = y_1 i + y_2 j + y_3 k \, .$$

Using the distributivity of multiplication over addition, the product becomes

$$q_1 \cdot q_2 = \vec{x} \cdot \vec{y} = (x_1 i + x_2 j + x_3 k)(y_1 i + y_2 j + y_3 k) =$$

$$= x_1 y_1 ii + x_1 y_2 ij + x_1 y_3 ik + x_2 y_1 ji + x_2 y_2 jj + x_2 y_3 jk +$$

$$+ x_3 y_1 ki + x_3 y_2 kj + x_3 y_3 kk .$$

It is quite long to understand what is going on. However, using Hamilton's Rules we can simplify the product of quaternions:

$$\vec{x} \cdot \vec{y} = -x_1 y_1 + x_1 y_2 k - x_1 y_3 j - x_2 y_1 k -$$

$$-x_2 y_2 + x_2 y_3 i + x_3 y_1 j - x_3 y_2 i - x_3 y_3 \, .$$

We regroup the terms according to the imaginary units, and we obtain

$$q_1 \cdot q_2 = \vec{x} \cdot \vec{y} = -(x_1 y_1 + x_2 y_2 + x_3 y_3) + (x_2 y_3 - x_3 y_2) i +$$

$$+ (x_3 y_1 - x_1 y_3) j + (x_1 y_2 - x_2 y_1) k \, .$$

Fortunately, we can utilize *the dot product*

$$x_1 y_1 + x_2 y_2 + x_3 y_3 = (\vec{x}, \vec{y})$$

and *the cross product*

$$(x_2 y_3 - x_3 y_2) i + (x_3 y_1 - x_1 y_3) j + (x_1 y_2 - x_2 y_1) k = \left[\vec{x}, \vec{y} \right]$$

of two vectors in $\mathbb{R}^3$ to write the above quaternion product in a more concise form

N.B.

$$q_1 \cdot q_2 = \vec{x} \cdot \vec{y} = -(\vec{x},\vec{y}) + [\vec{x},\vec{y}],$$

where $(\vec{x},\vec{y})$ is the scalar part, $\left[\vec{x},\vec{y}\right]$ is the imaginary part of the product of two imaginary quaternions.

Corollary

> *The product of two imaginary quaternions*
>
> $$q_1 = \vec{x} \quad \textit{and} \quad q_2 = \vec{y}$$
>
> *is also a quaternion, the real part of which is the dot product with the opposite sign, and the vector part is the cross product of these quaternions.*

Let's consider now the multiplication of two quaternions:

$$q_1 = x_0 + x_1 i + x_2 j + x_3 k = x_0 + \vec{x}\,,$$

$$q_2 = y_0 + y_1 i + y_2 j + y_3 k = y_0 + \vec{y}\,.$$

We get the rule of multiplication of quarterions

$$q_1 \cdot q_2 = (x_0 + \vec{x})(y_0 + \vec{y}) = x_0 y_0 + x_0 \vec{y} + y_0 \vec{x} - (\vec{x},\vec{y}) + [\vec{x},\vec{y}]\,,$$

or

$$q_1 \cdot q_2 = \{x_0,\, \vec{x}\} \cdot \{y_0,\, \vec{y}\} = \{x_0 y_0 - (\vec{x},\vec{y}),\, x_0 \vec{y} + y_0 \vec{x} + [\vec{x},\vec{y}]\}.$$

We see that the product of two quaternions is still a quaternion with scalar part

$$x_0 y_0 - (\vec{x},\vec{y})$$

and vector part

$$x_0 \vec{y} + y_0 \vec{x} + [\vec{x},\vec{y}]\,.$$

Example

> *Let's multiplicate quaternions*
>
> $$q_1 = 2 + 3i - j + 4k\,,$$
>
> $$q_2 = 4 + i + 2j - 3k\,.$$

❍ We first single out vector parts of the quaternions:

$$\vec{x} = (3,-1,4)\,, \quad \vec{y} = (1,2-3)\,.$$

Let us find the dot product and the cross product of these vectors:

$$(\vec{x},\vec{y}) = 3 - 2 - 12 = -11\,,$$

$$[\vec{x},\vec{y}] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 4 \\ 1 & 2 & -3 \end{vmatrix} = (-5)\vec{i} + 13\vec{j} + 7\vec{k}\ .$$

And we get

$$q_1 \cdot q_2 = 8 - (-11) + 2(i + 2j - 3k) + 4(3i - j + 4k) + (-5i + 13j + 7k) =$$

$$= 19 + 9i + 13j + 17k\,.$$

●

It is not difficult to verify that multiplication of quaternions is associative and distributes over addition:

$$(q_1 \cdot q_2) \cdot q_3 = q_1 \cdot (q_2 \cdot q_3)\,,$$

$$(q_1 + q_2) \cdot q_3 = q_1 \cdot q_3 + q_2 \cdot q_3\,,$$

$$q_1 \cdot (q_2 + q_3) = q_1 \cdot q_2 + q_1 \cdot q_3)\,.$$

Quaternion Conjugate

As in the case of complex number, we can define the conjugate of a quaternion.

Let

$$q = x_0 + x_1 i + x_2 j + x_3 k = x_0 + \vec{x}$$

be a quaternion. *The conjugate of* q, denoted as $\overline{q}$, is defined as

$$\overline{q} = x_0 - x_1 i - x_2 j - x_3 k = x_0 - \vec{x}\ .$$

Given two quaternions q_1 and q_2, we can easily verify that

$$\overline{q_1 + q_2} = \overline{q}_1 + \overline{q}_2 , \qquad \overline{q_1 \cdot q_2} = \overline{q}_2 \cdot \overline{q}_1 .$$

► Let's check, for example, the second formula.
Indeed, we have

$$q_1 \cdot q_2 = x_0 y_0 - (\vec{x}, \vec{y}) + x_0 \vec{y} + y_0 \vec{x} + [\vec{x}, \vec{y}]$$

and

$$\overline{q_1 \cdot q_2} = x_0 y_0 - (\vec{x}, \vec{y}) - x_0 \vec{y} - y_0 \vec{x} - [\vec{x}, \vec{y}] . \tag{3.7}$$

Then, let's find the product $\overline{q}_2 \cdot \overline{q}_1$:

$$\overline{q}_2 \cdot \overline{q}_1 = (y_0 - \vec{y})(x_0 - \vec{x}) = x_0 y_0 - (\vec{x}, \vec{y}) - x_0 \vec{y} - y_0 \vec{x} + [\vec{y}, \vec{x}] .$$

As the cross product of two vectors is anti-commutative, we have

$$\overline{q}_2 \cdot \overline{q}_1 = x_0 y_0 - (\vec{x}, \vec{y}) - x_0 \vec{y} - y_0 \vec{x} - [\vec{x}, \vec{y}] . \tag{3.8}$$

Comparing (3.7) and (3.8), we obtain

$$\overline{q_1 \cdot q_2} = \overline{q}_2 \cdot \overline{q}_1 . \quad ◄$$

From the definition we immediately have that the sum of the conjugate quarterions is the real number:

$$q + \overline{q} = 2x_0 .$$

Let's find the product of conjugate quaternions

$$q \cdot \overline{q} = (x_0 + \vec{x}) \cdot (x_0 - \vec{x}) = x_0^2 - \vec{x}^2 ,$$

$$\vec{x}^2 = \vec{x} \cdot \vec{x} = (x_1 i + x_2 j + x_3 k) \cdot (x_1 i + x_2 j + x_3 k) =$$

$$= x_1^2 ii + x_1 x_2 ij + x_1 x_3 ik + x_1 x_2 ji + x_2^2 jj + x_2 x_3 jk + x_1 x_3 ki + x_2 x_3 kj + x_3^2 kk =$$

$$= -x_1^2 - x_2^2 - x_3^2 = -(x_1^2 + x_2^2 + x_3^2).$$

Thus,

$$q \cdot \overline{q} = x_0^2 + x_1^2 + x_2^2 + x_3^2 .$$

N.B. *The product of conjugate quaternions is the positive real number which is equal to the sum of the squares of their coordinates.*

By analogy with the complex numbers let's call the number

$$\sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$$

as *modulus* of a quaternion q and denote it as $\left|q\right|$:

$$\left|q\right| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}\,.$$

Number

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 \overset{\text{den}}{=} \left\|q\right\|$$

is called the *norm* of a quaternion.

Thus, the product of conjugated quaternions is the square of the modulus or a norm of a quaternion:

$$q \cdot \overline{q} = \left|q\right|^2 = \left\|q\right\|.$$

A quaternion is called *a unit quaternion* if its modulus (norm) is equal to one. Any quaternion q can be normalized by dividing by its norm, to obtain a unit quaternion.

The *inverse* of a non-zero quaternion q is defined as

$$q^{-1} = \frac{\overline{q}}{\left|q\right|^2}\,.$$

Really, as we know

$$q \cdot q^{-1} = \{1, \vec{0}\} = 1\,.$$

We can easily verify that

$$q \cdot q^{-1} = q^{-1} \cdot q = 1\,.$$

In the case when q is *a unit quaternion*, the inverse is its conjugate $\overline{q}$:

$$q^{-1} = \overline{q}\,.$$

Remark

Of interest are two more operations with quaternions. First one is called a scalar product of quaternions and is expressed by the formula:

$$\frac{\overline{q}_1 \cdot q_2 + \overline{q}_2 \cdot q_1}{2} = \{x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3, \vec{0}\}\,.$$

As a result of this operation, a purely scalar quaternion is obtained, the scalar part of which is equal to the scalar product of quaternion-multipliers, if consider them as four-dimensional vectors, and the vector part is zero.

The second operation is called a vector cross product of quaternions:

$$\frac{q_1 \cdot q_2 - q_2 \cdot q_1}{2} = \{0, [\vec{x}, \vec{y}]\}\,.$$

As a result of this operation, a purely vector quaternion is obtained, the scalar part of which is equal to zero, and the vector part is equal to the vector product of vector parts of quaternion multipliers.

Euler's Identity

As in the case of real numbers for quaternions, the following theorem is valid.

Theorem 3.1.

The norm of the product of two quaternions q_1 and q_2 is the product of the individual norms.

► As we know,

$$\|q\| = q \cdot \overline{q} \quad \text{and} \quad \overline{q_1 \cdot q_2} = \overline{q}_2 \cdot \overline{q}_1\,,$$

then we have

$$\|q_1 \cdot q_2\| = (q_1 \cdot q_2)(\overline{q_1 \cdot q_2}) = (q_1 \cdot q_2)(\overline{q}_2 \cdot \overline{q}_1) = q_1 \cdot (q_2 \cdot \overline{q}_2) \cdot \overline{q}_1 =$$

$$\|q_1 \cdot q_2\| = q_1 \cdot (q_2 \cdot \overline{q}_2) \cdot \overline{q}_1 = q_1 \cdot \|q_2\| \cdot \overline{q}_1 = \|q_2\| \cdot q_1 \cdot \overline{q}_1 = \|q_2\| \cdot \|q_1\|\,.$$

Thus,

$$\left\| q_1 \cdot q_2 \right\| = \left\| q_1 \right\| \cdot \left\| q_2 \right\|$$

or we may write

$$\left| q_1 \cdot q_2 \right| = \left| q_1 \right| \left| q_2 \right| . \qquad \blacktriangleleft$$

Let us express this equality in components of quaternions. Consider two quaternions

$$q_1 = x_0 + x_1 i + x_2 j + x_3 k ,$$

$$q_2 = y_0 + y_1 i + y_2 j + y_3 k .$$

We have

$$q_1 \cdot q_2 = (x_0 + x_1 i + x_2 j + x_3 k)(y_0 + y_1 i + y_2 j + y_3 k) =$$

$$= (x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3) + i(x_0 y_1 + x_1 y_0 + x_2 y_3 - x_3 y_2) +$$

$$+ j(x_2 y_0 + x_0 y_2 + x_3 y_1 - x_1 y_3) + k(x_0 y_3 + x_3 y_0 + x_1 y_2 - x_2 y_1) .$$

Remember that

$$\left| q_1 \right|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 , \quad \left| q_2 \right|^2 = y_0^2 + y_1^2 + y_2^2 + y_3^2 .$$

Therefore

$$\left| q_1 \cdot q_2 \right|^2 = (x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3)^2 + (x_0 y_1 + x_1 y_0 + x_2 y_3 - x_3 y_2)^2 +$$

$$+ (x_2 y_0 + x_0 y_2 + x_3 y_1 - x_1 y_3)^2 + (x_0 y_3 + x_3 y_0 + x_1 y_2 - x_2 y_1)^2 .$$

And we get

$$(x_0^2 + x_1^2 + x_2^2 + x_3^2)(y_0^2 + y_1^2 + y_2^2 + y_3^2) =$$

$$= (x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3)^2 + (x_0 y_1 + x_1 y_0 + x_2 y_3 - x_3 y_2)^2 +$$

$$+ (x_2 y_0 + x_0 y_2 + x_3 y_1 - x_1 y_3)^2 + (x_0 y_3 + x_3 y_0 + x_1 y_2 - x_2 y_1)^2$$

— *the product of the sum of four squares by the sum of four squares is again a sum of four squares.*

This is well-known *Euler's Identity*.

3.2.2. Polar Form of Quaternion

Quaternions, like complex numbers, can be represented in polar form.

Consider quaternion

$$q = x_0 + \vec{x} \tag{3.9}$$

where x_0 and $\vec{x}$ are the scalar and vector parts, respectively.

Let us divide both parts of equality (3.9) by

$$|q| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}\,.$$

We get

$$\frac{q}{|q|} = \frac{x_0}{|q|} + \frac{\vec{x}}{|q|}\,.$$

The term $\dfrac{\vec{x}}{|x|}$ we represent as

$$\frac{\vec{x}}{|q|} = \frac{\vec{x}}{|q|} \cdot \frac{|\vec{x}|}{|\vec{x}|} = \frac{\vec{x}}{|\vec{x}|} \cdot \frac{|\vec{x}|}{|q|},$$

here $\dfrac{\vec{x}}{|\vec{x}|} = \vec{u}$ is a unit vector of vector-quaternion $\vec{x}$, and

$|\vec{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the modulus of $\vec{x}$.

Then we have

$$q = |q|\left(\frac{x_0}{|q|} + \frac{|\vec{x}|}{|q|}\vec{u}\right).$$

We introduce the notation now

$$\frac{x_0}{|q|} = \cos\varphi, \qquad \frac{\vec{x}}{|q|} = \sin\varphi\,.$$

Obviously, the well-known equality holds:

$$\cos^2\varphi + \sin^2\varphi = 1\,.$$

Indeed,

$$\cos^2\varphi+\sin^2\varphi=\left(\frac{x_0}{|q|}\right)^2+\left(\frac{\vec{x}}{|q|}\right)^2=\frac{x_0^2+x_1^2+x_2^2+x_3^2}{x_0^2+x_1^2+x_2^2+x_3^2}=1\,.$$

Thus, we obtain the *polar form* of quaternion

$$q=|q|(\cos\varphi+\vec{u}\sin\varphi)$$

that both in essence and in form is similar to the polar form of a complex number.

3.2.3. Matrix Model of Quaternion

We'll now use matrices to give representation of quaternion.

Let us consider two quaternions:

$$q_1=x_0+x_1i+x_2j+x_3k\,,$$

$$q_2=y_0+y_1i+y_2j+y_3k\,.$$

We may represent the quaternions as vectors in $\mathbb{R}^4$:

$$q_1=\begin{pmatrix}x_0\\x_1\\x_2\\x_3\end{pmatrix}\quad\text{and}\quad q_2=\begin{pmatrix}y_0\\y_1\\y_2\\y_3\end{pmatrix}$$

To discover a matrix form of a quaternion let's use quaternion multiplication.

As we know

$$q_1\cdot q_2=x_0y_0-(\vec{x},\vec{y})+x_0\vec{y}+y_0\vec{x}+[\vec{x},\vec{y}]$$

or in vector form

$$q_1\cdot q_2=\begin{pmatrix}x_0y_0-x_1y_1-x_2y_2-x_3y_3\\x_0y_1+y_0x_1+x_2y_3-x_3y_2\\x_0y_2+y_0x_2+x_3y_1-x_1y_3\\x_0y_3+y_0x_3+x_1y_2-x_2y_1\end{pmatrix}.$$

The latter expression is best represented as

$$q_1 \cdot q_2 = \begin{pmatrix} x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 \\ x_1y_0 + x_0y_1 - x_3y_2 + x_2y_3 \\ x_2y_0 + x_3y_1 + x_0y_2 - x_1y_3 \\ x_3y_0 - x_2y_1 + x_1y_2 + x_0y_3 \end{pmatrix}.$$

and it can be recognized as a matrix – vector product

$$q_1 \cdot q_2 = \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Here the first matrix is considered as coefficient matrix, the second matrix is quaternion q_2.

Thus the product of quaternion q_1 and quaternion q_2 we may write as the pair of two matrices

$$q_1 \cdot q_2 = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

We may assume that the coefficient matrix is *the matrix form* of quaternion q_1. This leads us to possibility of modeling quaternion by special square matrices of the fourth order:

$$q = x_0 + x_1 i + x_2 j + x_3 k = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix}.$$

The matrix

$$\mathrm{Q} = \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix}$$

has some very interesting properties.

The transpose of this matrix is the matrix

$$\mathrm{Q}^{\mathrm{T}} = \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ -x_1 & x_0 & x_3 & -x_2 \\ -x_2 & -x_3 & x_0 & x_1 \\ -x_3 & x_2 & -x_1 & x_0 \end{pmatrix} \mapsto \overline{q} \; .$$

We get the *matrix representation of quaternion conjugate.*

Let's multiply

$$\mathrm{Q} \cdot \mathrm{Q}^{\mathrm{T}} = \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix} \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix}^{\mathrm{T}} .$$

We obtain

$$\mathrm{Q} \cdot \mathrm{Q}^{\mathrm{T}} = \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix} \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ -x_1 & x_0 & x_3 & -x_2 \\ -x_2 & -x_3 & x_0 & x_1 \\ -x_3 & x_2 & -x_1 & x_0 \end{pmatrix} =$$

$$= \begin{pmatrix} x_0^2 + x_1^2 + x_2^2 + x_3^2 & 0 & 0 & 0 \\ 0 & x_0^2 + x_1^2 + x_2^2 + x_3^2 & 0 & 0 \\ 0 & 0 & x_0^2 + x_1^2 + x_2^2 + x_3^2 & 0 \\ 0 & 0 & 0 & x_0^2 + x_1^2 + x_2^2 + x_3^2 \end{pmatrix} =$$

$$= (x_0^2 + x_1^2 + x_2^2 + x_3^2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (x_0^2 + x_1^2 + x_2^2 + x_3^2) \cdot \mathrm{I}_4 \cdot$$

Thus, the matrix Q is *orthogonal matrix* if

$$\sum_{m=0}^{3} x_m^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \, .$$

In this representation, the fourth power of the modulus of a quaternion is the determinant of the corresponding matrix:

$$\det \mathrm{Q} = \begin{vmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{vmatrix} = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^2 = \left|q\right|^4 = \left\|q\right\|^2 .$$

Let’s find the matrix representation for very important quaternions:

identical quaternion

$$q = 1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathrm{I},$$

where I is our well-known Identity matrix;

$$q = i = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

$$q = j = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix};$$

$$q = k = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We may verify that all Hamilton's Rules holds in this case.

Indeed,

$$i^2 \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -\mathrm{I} \mapsto -1\,.$$

$$i \cdot j \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \mapsto k\,;$$

$$i \cdot j \cdot k \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -\mathrm{I} \mapsto -1.$$

We leave it as an exercise to verify other Hamilton's Rules for interested reader.

So any quaternion $q = x_0 + x_1 i + x_2 j + x_3 k$ we may represent in the following *matrix form*:

$$q = x_0 + x_1 i + x_2 j + x_3 k = x_0 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} +$$

$$+ x_1 \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Remark

This matrix representation of quaternion is not the only one.