3.3. Hypercomplex (Grassman) Numbers

– *Definition of Hypercomplex (Grassmann) Number*
– *Basic operations*
– *Hypercomplex (Grassmann) Numbers and Determinant*

In previous lectures, we considered complex numbers and quaternions. The starting point for the creation of new numerical systems are the laws of arithmetic. However, introducing *complex numbers*, we had to abandon the most important property of real numbers – *order*, i.e. the order symbols <, ≤, >, and ≥ , as we know, are not used with complex numbers. When creating another numerical system – *quaternions*, which already contains not one but three imaginary units, i.e. they are a four dimensional extension of complex numbers, we had to abandon a very important property – the *commutative multiplication* of two quaternions. The other laws of arithmetic are valid here, including such an important operation as division.

The result of the development of the theory of complex numbers is given by F. Frobenius' theorem: *among all numerical systems that satisfy the commutative and associative properties of addition, distributive and associative conditions of multiplication, the divisibility rule by a nonzero element holds for only three numerical systems: real numbers, complex numbers and quaternions*.

Hermann Grassmann

New attempts to generalize the concept of complex numbers led in the XIX century to the creation of new algebras. In his work "*Theory of Extensive Magnitudes*", Hermann Günter Grassmann developed the original calculus, which is associated with geometric representations.

In this lecture we will give only the information that is necessary to present the *theory of determinants*.

3.3.1. Basic Definitions

The complex number

$$z = x + iy$$

we may consider as a linear combination of the form

$$z = x \cdot 1 + y \cdot i,$$

where x, y is the ordered pair of real numbers; 1 and i form the basis; imaginary unit i satisfies the condition

$$i^2 = -1.$$

Quaternion

$$q = x_0 + x_1 i + x_2 j + x_3 k$$

we may consider as a linear form

$$q = x_0 \cdot 1 + x_1 \cdot i + x_2 \cdot j + x_3 \cdot k,$$

here x_0, x_1, x_2, x_3 are the ordered four real numbers; 1, i, j, k form basis, moreover 1 is the unit in algebra of quarternions, imaginary units i, j, k satisfy the *Hamilton's Rules*

$$i^2 = j^2 = k^2 = -1,$$

$$i \cdot j = k,\ j \cdot k = i,\ k \cdot i = j,$$

$$i \cdot j = -j \cdot i, \quad j \cdot k = -k \cdot j, \quad k \cdot i = -i \cdot k,$$

$$i \cdot j \cdot k = -1.$$

Complex numbers and quaternions, considered by us, are covered by a more general concept of hypercomplex system of numbers.

Definition 3.3.

Hypercomplex Number (*Grassmann Number*) is called a linear combination of n different imaginary units $e_1, e_2, \dots, e_n$:

$$H = x_1 e_1 + x_2 e_2 + \dots + x_n e_n ,$$

where the real numbers $x_1, x_2, \dots, x_n$ *are called coordinates of hypercomplex number;*

the imaginary units $e_1, e_2, \dots, e_n$ *satisfy the following conditions (Grassmann's Rules):*

$$e_i \cdot e_j = -e_j \cdot e_i ,$$

$$(e_i \cdot e_j) \cdot e_k = e_i \cdot (e_j \cdot e_k), \qquad (3.10)$$

$$e_i \cdot (e_j + e_k) = e_i \cdot e_j + e_i \cdot e_k .$$

Let's consider the main properties of hypercomplex Grassmann numbers. Due to hypercomplex numbers are an extension of complex numbers, many properties of them are already familiar.

Equality, Addition, Scalar Multiplication of Matrices

Two Grassman numbers

$$H_1 = x_1 e_1 + x_2 e_2 + \dots + x_n e_n = \sum_{i=1}^{n} x_i e_i$$

and

$$H_2 = y_1 e_1 + y_2 e_2 + \dots + y_n e_n = \sum_{i=1}^{n} y_i e_i$$

are defined to be *equal*, if their corresponding coordinates are equal:

$$H_1 = H_2 \Leftrightarrow \begin{cases} x_1 = y_1, \\ x_2 = y_2, \\ \quad \cdots \\ x_n = y_n. \end{cases}$$

Addition and *substraction* of two Grassmann numbers act similar to complex numbers — component-wise:

$$H_1 \pm H_2 = \sum_{i=1}^{n} x_i e_i \pm \sum_{i=1}^{n} y_i e_i = \sum_{i=1}^{n} (x_i \pm y_i) e_i .$$

If $H = \sum_{i=1}^{n} x_i e_i$ is *a* Grassmann Number and $\alpha \in \mathbb{R}$ is a scalar, then a *Scalar Multiple* of H by α is called the Grassmann Number

$$\alpha H = \sum_{i=1}^{n} (\alpha x_i) e_i .$$

Example

Given

$$H_1 = 2e_1 + 3e_2 - e_3 - 5e_4$$

and

$$H_2 = -3e_1 + 2e_2 + e_3 - e_4 .$$

Let's find $H_1 + 2H_2$ *and* $2H_1 - 3H_2$.

❍ According to the rules of addition and multiplication by real numbers we get:

$$H_1 + 2H_2 = (2-6)e_1 + (3+4)e_2 + (-1+2)e_3 + (-5-2)e_4 \Rightarrow$$

$$h_1 + 2h_2 = -4e_1 + 7e_2 + e_3 - 7e_4 .$$

$$2H_1 - 3H_2 = (4+9)e_1 + (6-6)e_2 + (-2-3)e_3 + (-10+3)e_4 \Rightarrow$$

$$2H_1 - 3H_2 = 13e_1 - 5e_3 - 7e_4 . \quad \bullet$$

Addition of Grassmann numbers and scalar multiplication of a Grassmann number by a scalar both have the following properties.

For Grassmann numbers H_1, H_2, H_3 *and scalars* $\alpha, \beta \in \mathbb{R}$ *we get:*

1) $H_1 + H_2 = H_2 + H_1$;

2) $H_1 + (H_2 + H_3) = (H_1 + H_2) + H_3$;

3) $H_1 + O = H_1$ $\quad (O = \sum_{i=1}^{n} 0 e_i$ is *Zero Grassmann number*);

4) $H_1 + (-H_1) = O$ $\quad ((-H_1 = (-1) \cdot H_1$ is the *Additive Inverse of Grassmann number* H_1);

5) $1 \cdot H_1 = H_1$;

6) $(\alpha + \beta) \cdot H_1 = \alpha \cdot H_1 + \beta \cdot H_1$;

7) $\alpha \cdot (H_1 + H_2) = \alpha \cdot H_1 + \alpha \cdot H_2$;

8) $\alpha \cdot (\beta \cdot H_1) = (\alpha\beta) \cdot H_1$.

Next let's take a look at multiplication of Grassmann numbers.

Grassmann Number Multiplication

The skew symmetric multiplication property of the product of the imaginary units $(e_i \cdot e_j = -e_j \cdot e_i)$ implies that *the product of the equal units is zero*, that is

N.B. ❗ $$e_i \cdot e_i = 0, \qquad i = 1, 2, \ldots, n. \qquad (3.11)$$

Indeed, when $i = j$ we obtain:

$$e_i \cdot e_i = -e_i \cdot e_i \Rightarrow 2(e_i \cdot e_i) = 0 \Rightarrow e_i \cdot e_i = 0.$$

Thus,

$$e_i \cdot e_j = \begin{cases} 0, & i = j, \\ -e_j \cdot e_i, i \neq j. \end{cases}$$

Multiplication of imaginary units has the following properties:

$$e_i \cdot (\alpha e_j + \beta e_k) = \alpha(e_i \cdot e_j) + \beta(e_i \cdot e_k),$$

$$(\alpha e_i + \beta e_j) \cdot e_k = \alpha(e_i \cdot e_k) + \beta(e_j \cdot e_k),$$

$$(\alpha e_i) \cdot (\beta e_j) = e_i \cdot (\alpha\beta) e_j = (\alpha\beta e_i) \cdot e_j = (\alpha\beta) e_i \cdot e_j.$$

N.B. *From the condition* (3.11), *it follows that the product of* n *imaginary units with two equal units is equal to zero.*

If we rearrange two imaginary units in the product of n units then this product will change by a factor of (-1).

For example,

$$e_i \cdot e_j \cdot e_k = -e_i \cdot e_k \cdot e_j = e_k \cdot e_i \cdot e_j = -e_k \cdot e_j \cdot e_i = e_j \cdot e_k \cdot e_i = -e_j \cdot e_i \cdot e_k.$$

Consider the general Grassmann numbers

$$H_k = \sum_{i=1}^{n} x_{ki} e_i = x_{k1} e_1 + x_{k2} e_2 + \ldots + x_{kn} e_n, \quad k = 1, 2, \ldots, n.$$

The imaginary units satisfy Grassmann's Rules, and so Grassmann numbers H_k inherit the same properties as the imaginary units, in particular:

1) $H_l \cdot H_m = -H_m \cdot H_l$,

2) $H_m \cdot H_m = 0$,

3) $H_l \cdot (\alpha H_m + \beta H_p) = \alpha(H_l \cdot H_m) + \beta(H_l \cdot H_p)$

4) $H_l \cdot (H_m \cdot H_p) = (H_l \cdot H_m) \cdot H_p = H_l \cdot H_m \cdot H_p$

Example

Let's multiply Grassmann numbers

$$H_1 = 2e_1 + 3e_2$$

and

$$H_2 = -3e_1 + 2e_2 .$$

❍ We multiply given Grassmann numbers as polynomials:

$$H_1 \cdot H_2 = (2e_1 + 3e_2) \cdot (-3e_1 + 2e_2) =$$

$$= -6e_1 \cdot e_1 + 4e_1 \cdot e_2 - 9e_2 \cdot e_1 + 6e_2 \cdot e_2 .$$

Since

$$e_1 \cdot e_1 = e_2 \cdot e_2 = 0, \; e_2 \cdot e_1 = -e_1 \cdot e_2,$$

we get

$$H_1 \cdot H_2 = 13e_1 \cdot e_2 . \quad \bullet$$

Example

Let's make sure that

$$H_1 \cdot H_2 = -H_2 \cdot H_1,$$

if

$$H_1 = 2e_1 - e_2$$

and

$$H_2 = e_2 + 3e_3 .$$

❍ We have

$$H_1 \cdot H_2 = (2e_1 - e_2) \cdot (e_2 + 3e_3) = 2e_1 \cdot e_2 - e_2 \cdot e_2 + 6e_1 \cdot e_3 - 3e_2 \cdot e_3 =$$

$$= 2e_1 \cdot e_2 + 6e_1 \cdot e_3 - 3e_2 \cdot e_3 .$$

And now let's find $H_2 \cdot H_1$:

$$H_2 \cdot H_1 = (e_2 + 3e_3) \cdot (2e_1 - e_2) = 2e_2 \cdot e_1 - e_2 \cdot e_2 + 6e_3 \cdot e_1 - 3e_3 \cdot e_2 =$$

$$= -2e_1 \cdot e_2 - 6e_1 \cdot e_3 + 3e_2 \cdot e_3 = -(2e_1 \cdot e_2 + 6e_1 \cdot e_3 - 3e_2 \cdot e_3).$$

Hence,

$$H_1 \cdot H_2 = -H_2 \cdot H_1 . \quad \bullet$$

Example

Let's multiply three Grassmann numbers

$$H_1 = 2e_1 + 3e_3,$$

$$H_2 = e_3 - 2e_2,$$

$$H_3 = e_1 + 3e_2 - e_3.$$

❍ We have

$$H_1 \cdot H_2 \cdot H_3 = (2e_1 + 3e_3) \cdot (e_3 - 2e_2) \cdot (e_1 + 3e_2 - e_3) =$$

$$= (2e_1 \cdot e_3 - 4e_1 \cdot e_2 + 3e_3 \cdot e_3 - 6e_3 \cdot e_2) \cdot (e_1 + 3e_2 - e_3).$$

Here $e_3 \cdot e_3 = 0$, and we continue to multiply our Grassmann numbers:

$$H_1 \cdot H_2 \cdot H_3 = (2e_1 \cdot e_3 - 4e_1 \cdot e_2 - 6e_3 \cdot e_2) \cdot (e_1 + 3e_2 - e_3) =$$

$$= 2e_1 \cdot e_3 \cdot e_1 + 6e_1 \cdot e_3 \cdot e_2 - 2e_1 \cdot e_3 \cdot e_3 - 4e_1 \cdot e_2 \cdot e_1 - 12e_1 \cdot e_2 \cdot e_2 +$$

$$+4e_1 \cdot e_2 \cdot e_3 - 6e_3 \cdot e_2 \cdot e_1 - 18e_3 \cdot e_2 \cdot e_2 + 6e_3 \cdot e_2 \cdot e_3.$$

Since the product of imaginary units with two equal units e_j is equal to zero, we obtain

$$H_1 \cdot H_2 \cdot H_3 = 6e_1 \cdot e_3 \cdot e_2 + 4e_1 \cdot e_2 \cdot e_3 - 6e_3 \cdot e_2 \cdot e_1.$$

After ordering e_j we get:

$$H_1 \cdot H_2 \cdot H_3 = 6e_1 \cdot e_3 \cdot e_2 + 4e_1 \cdot e_2 \cdot e_3 - 6e_3 \cdot e_2 \cdot e_1 =$$

$$= 6(-e_1 \cdot e_2 \cdot e_3) + 4e_1 \cdot e_2 \cdot e_3 - 6(-e_1 \cdot e_2 \cdot e_3) =$$

$$= 4e_1 \cdot e_2 \cdot e_3. \quad \bullet$$

Let's analyze the considered examples.

We first multiplied Grassmann numbers

$$H_1 = 2e_1 + 3e_2 \quad \text{and} \quad H_2 = -3e_1 + 2e_2 .$$

And we got

$$H_1 \cdot H_2 = 13e_1 \cdot e_2 .$$

The imaginary units $e_1, e_2, \ldots, e_n$ are often called the *first order basis elements*. The imaginary unit $e_{ij} \equiv e_i \cdot e_j$ is called a *second order basis element*. There exist C_n^2 different second order basis elements. We may also define the basis element of the *third order*

$$e_i \cdot e_j \cdot e_k \overset{den}{=} e_{ijk} ,$$

of the *fourth order* and etc. inductively.

So we may represent the product $H_1 \cdot H_2$ as

$$H_1 \cdot H_2 = 13e_{12} .$$

Let us form the matrix from the coordinates of the given hypercomplex numbers:

$$\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix},$$

and the determinant of this matrix is equal, as we know, to

$$\begin{vmatrix} 2 & 3 \\ -3 & 2 \end{vmatrix} = 13 .$$

And what do we discover? *The coefficient at the second order basis element coincides with the determinant.*

Perhaps this is a coincidence?

Let's take a look at multiplication of three Grassmann numbers:

$$H_1 = 2e_1 + 3e_3, \quad H_2 = e_3 - 2e_2, \quad H_3 = e_1 + 3e_2 - e_3 .$$

We obtained that

$$H_1 \cdot H_2 \cdot H_3 = 4e_1 \cdot e_2 \cdot e_3 = 4e_{123}.$$

We again construct the matrix of coordinates of Grassmann numbers

$$\begin{pmatrix} 2 & 0 & 3 \\ 0 & -2 & 1 \\ 1 & 3 & -1 \end{pmatrix}$$

and calculate its determinant

$$\begin{vmatrix} 2 & 0 & 3 \\ 0 & -2 & 1 \\ 1 & 3 & -1 \end{vmatrix} = 4 + 6 - 6 = 4\,.$$

Surprisingly, we got the same result.

So let's take a look at the next paragraph.

3.3.2. Hypercomplex Numbers and Determinant

Consider a square matrix A_n

$$\mathrm{A}_n = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

Let's introduce n Grassmann numbers associated to the rows of the matrix A_n

$$H_1 = a_{11}e_1 + a_{12}e_2 + \dots + a_{1n}e_n,$$

$$H_2 = a_{21}e_1 + a_{22}e_2 + \dots + a_{2n}e_n,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$H_n = a_{n1}e_1 + a_{n2}e_2 + \dots + a_{nn}e_n\,.$$

Consider the product of Grassmann numbers $H_1 \cdot H_2 \cdot ... \cdot H_n$ now.

This multiplication involves n^n terms. Some terms equal zero since they contain multiple factors of imaginary units. Some components will be non-zero due to the fact that they consist of the product of different imaginary units $e_1, e_2, ..., e_n$. Rearranging the adjacent multipliers and considering the skew symmetric multiplication property of the product of the imaginary units we can make the product of different imaginary units take the form $\pm e_1 \cdot e_2 \cdot ... \cdot e_n$. Finally after reduction of all the terms in this manner we arrive at the equality

$$H_1 \cdot H_2 \cdot ... \cdot H_n = \mathrm{D} e_1 \cdot e_2 \cdot ... \cdot e_n = \mathrm{D} e_{12...n}. \qquad (3.12)$$

The number D is our well-known determinant of matrix A_n

$$\mathrm{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Let's make sure that the well-known formulas for calculating the determinants of second and third orders are fulfilled even with this approach.

Consider the multiplication of two Grassmann numbers:

$$H_1 = a_{11}e_1 + a_{12}e_2 , \quad H_2 = a_{21}e_1 + a_{22}e_2.$$

For the product $H_1 \cdot H_2$ we have

$$H_1 \cdot H_2 = (a_{11}e_1 + a_{12}e_2) \cdot (a_{21}e_1 + a_{22}e_2) =$$

$$= a_{11}a_{21}e_1 \cdot e_1 + a_{11}a_{22}e_1 \cdot e_2 + a_{12}a_{21}e_2 \cdot e_1 + a_{12}a_{22}e_2 \cdot e_2 =$$

$$= (a_{11}a_{22} - a_{12}a_{21})e_1 \cdot e_2 = (a_{11}a_{22} - a_{12}a_{21})e_{12}.$$

Thus, we obtain

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

This formula is familiar to us.

Consider the multiplication of three Grassmann numbers:

$$H_1 = a_{11}e_1 + a_{12}e_2 + a_{13}e_3 \,, \quad H_2 = a_{21}e_1 + a_{22}e_2 + a_{23}e_3,$$

$$H_3 = a_{31}e_1 + a_{32}e_2 + a_{33}e_3.$$

While calculating the product $H_1 \cdot H_2 \cdot H_3$ the majority of the terms are equal to zero since they contain a product of equal units. So we'll only write down those terms that contain the product of the different imaginary units e_1, e_2, e_3. Thus we have

$$H_1 \cdot H_2 \cdot H_3 = a_{11}a_{22}a_{33}e_{123} + a_{11}a_{23}a_{32}e_{132} + a_{13}a_{21}a_{33}e_{213} +$$

$$+a_{12}a_{23}a_{31}e_{231} + a_{13}a_{21}a_{32}e_{312} + a_{13}a_{22}a_{31}e_{321}.$$

Using the following equalities

$$e_{123} = -e_{132} = -e_{213} = e_{231} = e_{312} = -e_{321},$$

we find that

$$H_1 \cdot H_2 \cdot H_3 = (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}e_{132} - a_{13}a_{21}a_{33}e_{213} +$$

$$+a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31})e_{123}.$$

So for the determinant of a third order we obtain a number:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} -$$

$$-a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

Thus, we get the formula already known.

We have got another method for calculating the determinant of the n-th order – polynomial method.

Example

Let's calculate the determinant

$$\begin{vmatrix} 2 & 5 & 0 & 2 \\ 3 & -1 & 2 & 1 \\ 4 & 0 & 1 & -2 \\ 3 & 1 & 0 & 1 \end{vmatrix}.$$

❍ Let us first introduce four Grassmann numbers associated to the rows of the matrix

$$A_4 = \begin{pmatrix} 2 & 5 & 0 & 2 \\ 3 & -1 & 2 & 1 \\ 4 & 0 & 1 & -2 \\ 3 & 1 & 0 & 1 \end{pmatrix}.$$

We have

$$H_1 = 2e_1 + 5e_2 + 2e_4, \qquad H_2 = 3e_1 - e_2 + 2e_3 + e_4,$$

$$H_3 = 4e_1 + e_3 - 2e_4, \qquad H_4 = 3e_1 + e_2 + e_4.$$

Try to multiply given Grassmann numbers:

$$H_1 \cdot H_2 \cdot H_3 \cdot H_4 = (2e_1 + 5e_2 + 2e_4) \cdot (3e_1 - e_2 + 2e_3 + e_4) \cdot$$

$$\cdot (4e_1 + e_3 - 2e_4) \cdot (3e_1 + e_2 + e_4).$$

Let us group the factors as follows

$$H_1 \cdot H_2 \cdot H_3 \cdot H_4 = (H_1 \cdot H_2) \cdot (H_3 \cdot H_4)$$

and multiply them.

We obtain

$$H_1 \cdot H_2 = (2e_1 + 5e_2 + 2e_4) \cdot (3e_1 - e_2 + 2e_3 + e_4) =$$

$$= -2e_{12} + 4e_{13} + 2e_{14} + 15e_{21} + 10e_{23} + 5e_{24} + 6e_{41} - 2e_{42} + 4e_{43} =$$

$$= -2e_{12} + 4e_{13} + 2e_{14} - 15e_{12} + 10e_{23} + 5e_{24} - 6e_{14} + 2e_{24} - 4e_{34} =$$

$$= -17e_{12} + 4e_{13} - 4e_{14} + 10e_{23} + 7e_{24} - 4e_{34}.$$

$$H_3 \cdot H_4 = (4e_1 + e_3 - 2e_4) \cdot (3e_1 + e_2 + e_4) =$$

$$= 4e_{12} + 4e_{14} + 3e_{31} + e_{32} + e_{34} - 6e_{41} - 2e_{42} =$$

$$= 4e_{12} + 4e_{14} - 3e_{13} - e_{23} + e_{34} + 6e_{14} + 2e_{24} =$$

$$= 4e_{12} - 3e_{13} + 10e_{14} - e_{23} + 2e_{24} + e_{34}.$$

Therefore,

$$H_1 \cdot H_2 \cdot H_3 \cdot H_4 =$$

$$= (-17e_{12} + 4e_{13} - 4e_{14} + 10e_{23} + 7e_{24} - 4e_{34}) \cdot$$

$$\cdot (4e_{12} - 3e_{13} + 10e_{14} - e_{23} + 2e_{24} + e_{34}) =$$

$$= -17e_{1234} + 8e_{1324} + 4e_{1423} + 100e_{2314} - 21e_{2413} - 16e_{3412}.$$

Since

$$e_{1324} = -e_{1234}, \quad e_{1423} = e_{1234},$$

$$e_{2314} = e_{1234}, \quad e_{2413} = -e_{1234},$$

$$e_{3412} = e_{1234},$$

we get

$$H_1 \cdot H_2 \cdot H_3 \cdot H_4 == (-17 + 8 + 4 + 100 + 21 - 16)e_{1234} = 84e_{1234}.$$

It means that

$$\begin{vmatrix} 2 & 5 & 0 & 2 \\ 3 & -1 & 2 & 1 \\ 4 & 0 & 1 & -2 \\ 3 & 1 & 0 & 1 \end{vmatrix} = 84\,.$$

❗ *And, cheers, we got the same result as in lecture* 3.

All well-known properties of determinants come directly from the equality (3.12).

Consider some of them.

Property 1

If all elements of a row of a matrix are equal to zero then the determinant of the matrix is equal to zero.

Indeed, if all elements $a_{ij} = 0$ then $H_i = 0$. Consequently,

$$H_1 \cdot H_2 \cdot \ldots \cdot H_n = 0 \Leftrightarrow \mathrm{D}e_{12\ldots n} = 0 \Rightarrow \mathrm{D} = 0.$$

If two rows of a determinant are interchanged, the sign of a determinant is reversed.

Property 2

Rearranging two rows i and j is equivalent to rearranging two factors H_i and H_j in the product $H_1 \cdot H_2 \cdot \ldots \cdot H_n$. Hence by skew symmetry

$$H_i \cdot H_j = -H_j \cdot H_i$$

the determinant changes its sign.

Property 3

If any two rows of a matrix are identical then the determinant of the matrix is equal to zero.

Indeed, if the product $H_1 \cdot H_2 \cdot \ldots \cdot H_n$ contains two or more similar factors, it equals, as we know, zero.

If one of the rows of the determinant is linearly expressed via the other rows then the determinant is equal zero.

Property 4

For example, let the first row of the determinant be a linear combination of the other ones, i.e.

$$H_1 = \alpha_2 H_2 + \ldots + \alpha_n H_n.$$

For the matrix elements we have the following equality:

$$a_{1j} = \alpha_2 a_{2j} + ... + \alpha_n a_{nj},\ j = \overline{1, n}.$$

Let's substitute H_1 into (3.12). We get

$$H_1 \cdot H_2 \cdot ... \cdot H_n = (\alpha_2 H_2 + ... + \alpha_n H_n) \cdot H_2 \cdot ... \cdot H_n =$$

$$= \alpha_2 H_2 \cdot H_2 \cdot ... \cdot H_n + ... + \alpha_n H_n \cdot H_2 \cdot ... \cdot H_n = 0.$$

Thus,

$$\mathrm{D} = 0.$$

Each term contains the product of Grassmann numbers with repeated factors, therefore, each term is equal to zero.

Property 5

If each element of a row can be expressed as binomial, the determinant can be written as the sum of two determinants.

Let each element of the i-th row of a matrix be composed of the sum of two elements:

$$H_i = H_i' + H_i''.$$

Then we have

$$H_1 \cdot H_2 \cdot ... \cdot (H_i' + H_i'') \cdot ... \cdot H_n =$$

$$= H_1 \cdot H_2 \cdot ... \cdot H_i' \cdot ... \cdot H_n + H_1 \cdot H_2 \cdot ... \cdot H_i'' \cdot ... \cdot H_n.$$

Q.E.D.

Other properties of determinants are proved similarly.

Questions to Chapter 3

I.

1. State the axiomatic definition of a Complex Number.
2. What is a Cartesian Complex number?
3. Give the geometric representation of a Complex number.
4. How can we operate with Cartesian Complex numbers?
5. What is a complex conjugate?
6. State the Euler's identity for Complex numbers and Quaternions.
7. What is a Polar Form of a Complex Number?
8. How can we multiply and divide Complex numbers in Polar form?
9. State the Moivre's formula.
10. State the Euler's formula.
11. What is an Exponential Form of a Complex Number?
12. What is a Matrix Model of a Complex Number?
13. What is called the modulus and the principal argument of a Complex Number?
14. What is a Quaternion?
15. State the structure of a Quaternion?
16. How we multiply the Quaternion?
17. What is a Quaternion conjugate?
18. What is a Polar form of a Quaternion?
19. What is a Matrix Model of a Quaternion?
20. What is a Grassmann number?
21. How can we operate with Grassmann numbers?
22. What are the properties of multiplication of imaginary units in Grassmann numbers?
23. State the structure of the product of Grassman numbers.

II.

1. Express complex numbers

$$z_1 = 1; \quad z_2 = i$$

in exponential form.

2. Express complex number

$$z = 1 + \cos 2 + i \sin 2$$

in polar form.

3. Find all roots $\sqrt[4]{-1}$ and depict them on the Complex plane.

4. Depict on the Complex Plane a set of points that satisfy the next conditions:

1) $\left|z + i\right| = \left|z - i\right|$;

2) $\arg(z - 1) = \dfrac{\pi}{3}$;

3) $0 \le \operatorname{Im} z < \operatorname{Re} z < 2$.

5. Is the number $-3i$ negative?

6. Solve the equation in the complex number system $z^6 + 16i = 0$.

7. Find the modulus and the argument of complex numbers

$$(1 - \sqrt{2})i; \qquad 1 - \sqrt{2}.$$

8. Find $\operatorname{Im} \overline{z}$ if $z = \dfrac{i}{1 - 2i}$.

9. Multiply two quaternions

$$q_1 = -1 + i + j - 2k, \quad q_2 = 2 - i + 3j - k.$$

10. Reduce the product $e_3 \cdot e_4 \cdot e_1 \cdot e_2$ of imaginary units to standard form.

11. Find the product of Grassmann numbers

$$H_1 = e_1 - 3e_2, \qquad H_2 = 3e_1 + 2e_2 - e_3, \qquad H_3 = e_1 + 4e_3.$$