Answers

1. Let

$$A = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}.$$

Calculate $A^T \cdot A^{-1}$.

Is A a symmetric matrix?

What is the trace of the transpose of $f(A)$, where

$$f(x) = x^2 - 1?$$

$(A^T \cdot A^{-1} = \frac{1}{5}\begin{pmatrix} 11 & -4 \\ -2 & 3 \end{pmatrix}$, $\mathrm{tr} f(A) = 9)$.

2. Find:

$$\text{a)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^n; \qquad \text{b)} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n.$$

(a) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^n = 2^{n-1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, b) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$).

3. Given the matrix

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Find $A^2,\ A^3, \ldots, A^n$.

$(A^n = \begin{pmatrix} \cos n\alpha & -\sin n\alpha \\ \sin n\alpha & \cos n\alpha \end{pmatrix})$

4. Find

$$\begin{pmatrix} 4 & 3 & -3 \\ 2 & 3 & -2 \\ 4 & 4 & -3 \end{pmatrix}^6$$

by using equality

$$\begin{pmatrix} 4 & 3 & -3 \\ 2 & 3 & -2 \\ 4 & 4 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & -1 \\ -2 & -5 & 4 \end{pmatrix}.$$

$$(\begin{pmatrix} 4 & 3 & -3 \\ 2 & 3 & -2 \\ 4 & 4 & -3 \end{pmatrix}^6 = \begin{pmatrix} 190 & 189 & -189 \\ 126 & 127 & -126 \\ 252 & 252 & -251 \end{pmatrix}:$$

$$\begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & -1 \\ -2 & -5 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}).$$

5. Given the matrix

$$A=\begin{pmatrix} 1 & -1 \\ 3 & -4 \end{pmatrix}.$$

Represent this matrix as a sum of symmetric and skew-symmetric matrices.

$$(A=\begin{pmatrix} 1 & -1 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}).$$

6. Compute the determinant

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix},$$

where α, β, γ are the roots of the equation

$$x^3 + px + q = 0.$$

(0).

7. Compute the determinant

$$\Delta = \begin{vmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & -3 & -8 \\ -1 & 1 & 0 & -13 \\ 2 & 3 & 5 & 15 \end{vmatrix}$$

by multiplying it by the determinant

$$\delta = \begin{vmatrix} 1 & -2 & -3 & -11 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

(24) ($\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & -3 & -8 \\ -1 & 1 & 0 & -13 \\ 2 & 3 & 5 & 15 \end{pmatrix}\begin{pmatrix} 1 & -2 & -3 & -11 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 3 & 3 & 0 \\ 2 & -1 & -1 & 4 \end{pmatrix}$).

8. Compute the determinant

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ -1 & 0 & -3 & \cdots & -n \\ -1 & -2 & 0 & \cdots & -n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & -3 & \cdots & 0 \end{vmatrix}$$

$(n!)$

9. Solve matrix equations

1) $\begin{pmatrix} 2 & -3 \\ 4 & -6 \end{pmatrix} \cdot \mathrm{X} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$; 2) $\mathrm{X} \cdot \begin{pmatrix} 3 & -5 \\ 9 & -15 \end{pmatrix} = \begin{pmatrix} 9 & -15 \\ 6 & -10 \end{pmatrix}$;

3) $\begin{pmatrix} 3 & -1 & 2 \\ 4 & -3 & 3 \\ 1 & 3 & 0 \end{pmatrix} \cdot \mathrm{X} = \begin{pmatrix} 3 & 9 & 7 \\ 1 & 11 & 7 \\ 7 & 5 & 7 \end{pmatrix}$.

$(\varnothing, \mathrm{X} = \begin{pmatrix} 3-3b & b \\ 2-3d & d \end{pmatrix})$.

10. Given the system

$$\begin{cases} x + \lambda y = 2, \\ x + (\lambda - 1)y - \lambda z = 2, \\ 2x + (\lambda - 2)y - \lambda z = 4 - \lambda. \end{cases}$$

1) Determine parameters $\lambda \in \mathbb{R}$ for which the system has:

a) exactly one solution;

b) infinitely many solutions;

c) no solutions.

2) Determine the general solution for λ's such that system has infinitely many solutions.

(1) a) $\lambda \neq 0, \lambda \neq -2$ b) $\lambda = 0$, c) $\lambda = -2$. 2) $(2, 0, c)$.

11. Compute determinant

$$\begin{vmatrix} x+1 & 1 & 0 & 2x & 1 \\ 2x & x+1 & 0 & x-1 & x+1 \\ 1 & 0 & 7 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ x+1 & x & 0 & 1 & x+1 \end{vmatrix}.$$

Determine for which $x \in \mathbb{R}$ the determinant is non-zero.

$(7x(1-x^2)$; $x \neq 0, x \neq \pm 1)$.

12. Given the following matrix equation

$$\frac{1}{5}(\mathrm{X}^{\mathrm{T}}\mathrm{A})^{\mathrm{T}} \cdot \mathrm{B}^{-1} = \mathrm{I},$$

where $\mathrm{A}, \mathrm{B}, \mathrm{X}$ are invertible 2×2 matrices and I is the Identity matrix.

1) Compute determinant $\det(\mathrm{X})$ assuming that

$$\det \mathrm{A} = 2 \det \mathrm{B};$$

2) Find X from this equation if

$$\mathrm{A} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, \qquad \mathrm{B} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}.$$

$(2{,}5;\ \ \mathrm{X} = \begin{pmatrix} -5 & -10 \\ 20 & 25 \end{pmatrix})$.

13. Given the equation

$$\mathrm{A}^2 + \mathrm{A} + \mathrm{I} = 0.$$

Prove that matrix A is non-singular matrix.

Specify a simple method for finding the inverse matrix to matrix A.

$(\mathrm{A}^{-1} = -(\mathrm{A} + \mathrm{I}))$.

14. Determine parameters α for which the matrix

$$A=\begin{pmatrix} \alpha & 4 & 1 \\ 2 & 5 & -1 \\ 0 & \alpha & 1 \end{pmatrix}$$

has no inverse matrix?

(-8;1)

15. Find the rank of the matrix

$$\begin{pmatrix} 1 & \lambda & -1 & 2 \\ 2 & -1 & \lambda & 5 \\ 1 & 10 & -6 & 1 \end{pmatrix},$$

for different values of parameter λ.

$(\lambda = 3 : r = 2; \lambda \neq 3, r = 3)$.

16. Find the coordinates of the point M in 3-dimension if its radius- vector forms equal angles with the coordinate axes, and the length of the radius- vector equals 3.

$(M_1(\sqrt{3};\sqrt{3};\sqrt{3}), M_2(-\sqrt{3};-\sqrt{3};-\sqrt{3}))$.

17. Given vectors $\vec{a}(2,-3,6)$ and $\vec{b}(-1,2,-2)$ with common initial point. Determine the coordinates of the vector $\vec{c}$, which is directed along the bisector of the angle between vectors $\vec{a}$ and $\vec{b}$, if $\left|\vec{c}\right| = 3\sqrt{42}$.

$(\vec{c}(-3,15,12))$

18. Find the horizontal and vertical components of the vector

$$\vec{a}(2,-1,3) \text{ on } \vec{b}(4,-1,2).$$

$(\vec{a}^{\|\vec{b}}(\frac{20}{7};-\frac{5}{7};\frac{10}{7});\vec{a}^{\perp\vec{b}}(-\frac{6}{7};-\frac{2}{7};\frac{11}{7}))$.

19. Consider two non-collinear vectors $\vec{a}$ and $\vec{b}$. Find constants α and β if the vectors

$$\vec{c} = \alpha\vec{a} + \beta\vec{b} \text{ and } \vec{d} = (\beta+1)\vec{a} + (2-\alpha)\vec{b}$$

are equal.

$(\alpha = 1,5, \beta = 0,5)$.

20. Consider three coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. Calculate the length of the vector

$$\vec{p} = \vec{a} + \vec{b} - \vec{c},$$

if

$$\left|\vec{a}\right| = 3, \ \left|\vec{b}\right| = 2, \ \left|\vec{c}\right| = 5, \ (\vec{a},\vec{b}) = (\vec{b},\vec{c}) = 60^\circ.$$

(7).

21. Find the coordinates of the vector $\vec{x}$ if it is known that

1) vector $\vec{x}$ is collinear to the vector $\vec{a}(6;-8;-7,5)$;

2) vector $\vec{x}$ forms an acute angle with the axis Oz ;

3) $\left|\vec{x}\right| = 50$.

$(\vec{x}(-24;32;30))$.

22. Find the projection of the vector $\vec{a}(\sqrt{2};-3;-5)$ onto the axis which forms with the coordinate axes Ox and Oz angles $\alpha = 45^\circ$, $\gamma = 60^\circ$ and the acute angle with the axis Oy.

(-3).

23. Find

$$\left|[(3\vec{a}+\vec{b}),(\vec{a}-3\vec{b})]\right|$$

if it is known that

$$\left|\vec{a}\right|=3,\ \left|\vec{b}\right|=5,\quad (\vec{a},\vec{b})=60^{\circ}.$$

$(75\sqrt{3})$.

24. What condition the vectors $\vec{a}$ and $\vec{b}$ satisfy so that the vectors

$$\vec{m}=3\vec{a}+\vec{b} \text{ and } \vec{n}=\vec{a}-3\vec{b}$$

are collinear ?

25. Calculate the area of the parallelogram which diagonals are given by the vectors

$$\vec{d}_1=3\vec{m}+\vec{n},\vec{d}_2=\vec{m}-5\vec{n},$$

if

$$\left|\vec{m}\right|=\left|\vec{n}\right|=1,\ (\vec{m},\vec{n})=45^{\circ}.$$

$(8\sqrt{2})$.

26. Find the vector $\vec{m}$ that is orthogonal to the axis Oz and to the vector $\vec{p}(8;-15;3)$, and forms the acute angle with the axis Ox, if $\left|\vec{m}\right|=51$.

$(\vec{m}(45;24;0)$

27. Define the type of the triplet $\vec{a}, \vec{b}, \vec{c}$ if

$$\vec{a}=\vec{j}-2\vec{k},\quad \vec{b}=\vec{i}+\vec{k},\quad \vec{c}=2\vec{i}.$$

(right).

28. Vector $\vec{c}$ is orthogonal to the vectors $\vec{a}$ and $\vec{b}$, the angle between vectors $\vec{a}$ and $\vec{b}$ equals 30° $((\vec{a},\vec{b}) = 30^{\circ})$,

$$|\vec{a}| = 6, \quad |\vec{b}| = 3, \quad |\vec{c}| = 3.$$

Find the triple scalar product $(\vec{a},\vec{b},\vec{c})$.

$((\vec{a},\vec{b},\vec{c}) = \pm 27)$.

29. Prove that the points

$$A(1;2;-1),\ B(0,1,5),\ C(-1,2,1),\ D(2,1,3)$$

lie in the same plane.

$(\overrightarrow{AB}(-1;-1;6), \overrightarrow{AC}(-2;0;2), \overrightarrow{AD}(1;-1;4), (\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}) = 0)$.

30. Given the matrix A

$$\mathrm{A} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

and the vector $\vec{x}$

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

1) Calculate all possible dot products between the vectors $\vec{x}$ and $\mathrm{A}\vec{x}$.
2) Compute the lengths of the vectors $\vec{x}$ and $\mathrm{A}\vec{x}$.
3) Find the angle between the vectors $\mathrm{A}\vec{x}$ and $\vec{x}$.
4) Draw a picture of these vectors in the plane.

$((x^2 + y^2)\cos\varphi; \sqrt{x^2 + y^2})$.

31. Prove the identity

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}).$$

32. Draw the line

$$\left|z-2\right| = \left|1-2\overline{z}\right|$$

on the Complex plane.

$(x^2 + y^2 = 1)$

33. Which line on the plane xOy is determined by the equation

$$z\overline{z} + i(z-\overline{z}) - 2 = 0\,?$$

$(x^2 + (y-1)^2 = 3)$

34. Solve the equation

$$z^4 - 1 + i = 0,$$

if it is known that

$$\operatorname{Re} z < 0,\ \operatorname{Im} z > 0.$$

($z = \dfrac{-1+i}{\sqrt[8]{8}}$).

35. Calculate

$$5(\cos 10^{\circ} + i\sin 10^{\circ}) \cdot 2(\cos 80^{\circ} + i\sin 80^{\circ})\,.$$

$(10i)$.

36. Find the modulus of the complex number

$$z = \frac{x^2 - y^2 + i2xy}{xy\sqrt{2} + i\sqrt{x^4 + y^4}}.$$

$(\left|z\right| = 1)$

37. Find the modulus $\left|z\right|$ and the argument $\arg z$ of the complex number

$$z = -3e^{\frac{\pi}{3}i}.$$

$(|z| = 3, \quad \arg z = -\frac{2}{3}\pi)$.

38. Find the angle by which you need to rotate the vector

$$3\sqrt{2} + i\sqrt{2}$$

to get the vector

$$-5 + i?$$

$(\varphi = \frac{3}{4}\pi)$.

39. Represent the complex number

$$1 - \cos\alpha + i\sin\alpha, \ 0 \le \alpha \le \pi,$$

in the Polar form.

$(z = 2\sin\frac{\alpha}{2}(\cos\frac{\pi - \alpha}{2} + i\sin\frac{\pi - \alpha}{2}))$.

40. Solve the equation

$$|z| - z = 1 + 2i.$$

$(z = \frac{3}{2} - 2i)$.

41. Calculate

$$z^{2020} + \frac{1}{z^{2020}},$$

if it is known that

$$z + \frac{1}{z} = 1.$$

(-1).

42. Given

$$(1+i)^n = (1-i)^n.$$

Find the integer number n.

$(n = 4k,\ k \in \mathbb{R})$.

43. Which of the following equations are the equations of the axis Ox:

$$z = 0,$$

$$z + \overline{z} = 0,$$

$$z = \overline{z},$$

$$\arg z = 0,$$

$$\operatorname{Im} z = 0,$$

$$|z - i| = |z + i|,$$

$$\arg z = \frac{\pi}{2},$$

$$\operatorname{Re} z = 0,$$

$$|z - 1| = |z + 1|?$$

$(z = \overline{z}, \operatorname{Im} z = 0, |z - i| = |z + i|)$.

44. Reduce the product of imaginary units

$$e_9 \cdot e_2 \cdot e_4 \cdot e_3 \cdot e_5 \cdot e_7 \cdot e_8 \cdot e_1 \cdot e_6$$

to the standard form.

$(e_1 \cdot e_2 \cdot e_3 \cdot e_4 \cdot e_5 \cdot e_6 \cdot e_7 \cdot e_8)$.

45. Compute the determinant

$$\begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{vmatrix}$$

by polynomial method.

Check the result by another known method.

(0).