Mid-Term Exam

You are suggested a number of problems to solve.

Problems are divided into three parts:

Part 1 "*Elements of Linear Algebra*": problems 1-15;

Part 2 "*Vector Algebra*": problems 16-31;

Part 3 " *Fundamentals of Grassmann Algebra*": problems 32-45.

Each part is evaluated separately.

For Part 1, the student can score a maximum of 80 points:

Problem 1 – 4 p.;

Problem 2 – 5 p.;

Problem 3 – 5 p.;

Problem 4 – 5 p.;

Problem 5 – 4 p.;

Problem 6 – 5 p.;

Problem 7 – 4 p.;

Problem 8 – 5 p.;

Problem 9 – 6 p.;

Problem 10 – 7 p.;

Problem 12 – 6 p.;

Problem 13 – 5 p.;

Problem 14 – 5 p.;

Problem 15 – 7 p..

The work is accepted as passed if the student scored 48 points.

For Part 2, the student can score a maximum of 70 points:

Problem 16 – 4 p.;

Problem 17 – 6 p.;

Problem 18 – 4 p.;

Problem 19 – 5 p.;

Problem 20 – 5 p.;

Problem 21 – 5 p.;

Problem 22 – 5 p.;

Problem 23 – 3 p.;

Problem 24 – 4 p.;

Problem 25 – 5 p.;

Problem 26 – 5 p.;

Problem 27 – 3 p.;

Problem 28 – 4 p.;

Problem 29 – 4 p.;

Problem 30 – 4 p.;

Problem 31 – 4 p..

The work is accepted as passed if the student scored 42 points.

For Part 3, the student can score a maximum of 65 points:

Problem 32 – 4 p.;

Problem 33 – 4 p.;

Problem 34 – 5 p.;

Problem 35 – 3 p.;

Problem 36 – 5 p.;

Problem 37 – 4 p.;

Problem 38 – 5 p.;

Problem 39 – 5 p.;

Problem 40 – 4 p.;

Problem 41 – 7 p.;

Problem 42 – 6 p.;

Problem 43 – 5 p.;

Problem 44 – 3 p.;

Problem 45 – 5 p.

The work is accepted as passed if the student scored 39 points.

The final mark depends on the amount of points, that the student scored.

Part 1.

1. Let

$$A = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}.$$

Calculate $A^T \cdot A^{-1}$.

Is A a symmetric matrix?

What is the trace of the transpose of $f(A)$, where

$$f(x) = x^2 - 1?$$

2. Find:

$$\text{a)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^n; \qquad \text{b)} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n.$$

3. Given the matrix

$$A=\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Find $A^2,\ A^3, \ldots, A^n$.

4. Find

$$\begin{pmatrix} 4 & 3 & -3 \\ 2 & 3 & -2 \\ 4 & 4 & -3 \end{pmatrix}^6$$

by using equality

$$\begin{pmatrix} 4 & 3 & -3 \\ 2 & 3 & -2 \\ 4 & 4 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & -1 \\ -2 & -5 & 4 \end{pmatrix}.$$

5. Given the matrix

$$A=\begin{pmatrix} 1 & -1 \\ 3 & -4 \end{pmatrix}.$$

Represent this matrix as a sum of symmetric and skew-symmetric matrices.

6. Compute the determinant

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix},$$

where α, β, γ are the roots of the equation

$$x^3 + px + q = 0.$$

7. Compute the determinant

$$\Delta=\begin{vmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & -3 & -8 \\ -1 & 1 & 0 & -13 \\ 2 & 3 & 5 & 15 \end{vmatrix}$$

by multiplying it by the determinant

$$\delta=\begin{vmatrix} 1 & -2 & -3 & -11 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

8. Compute the determinant

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ -1 & 0 & -3 & \cdots & -n \\ -1 & -2 & 0 & \cdots & -n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & -3 & \cdots & 0 \end{vmatrix}.$$

9. Solve matrix equations

1) $\begin{pmatrix} 2 & -3 \\ 4 & -6 \end{pmatrix} \cdot \mathrm{X} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix};$ 2) $\mathrm{X} \cdot \begin{pmatrix} 3 & -5 \\ 9 & -15 \end{pmatrix} = \begin{pmatrix} 9 & -15 \\ 6 & -10 \end{pmatrix};$

$$3)\ \begin{pmatrix} 3 & -1 & 2 \\ 4 & -3 & 3 \\ 1 & 3 & 0 \end{pmatrix} \cdot \mathrm{X} = \begin{pmatrix} 3 & 9 & 7 \\ 1 & 11 & 7 \\ 7 & 5 & 7 \end{pmatrix}.$$

10. Given the system

$$\begin{cases} x + \lambda y = 2, \\ x + (\lambda - 1)y - \lambda z = 2, \\ 2x + (\lambda - 2)y - \lambda z = 4 - \lambda. \end{cases}$$

1) Determine parameters $\lambda \in \mathbb{R}$ for which the system has:

a) exactly one solution;

b) infinitely many solutions;

c) no solutions.

2) Determine the general solution for λ's such that system has infinitely many solutions.

11. Compute determinant

$$\begin{vmatrix} x+1 & 1 & 0 & 2x & 1 \\ 2x & x+1 & 0 & x-1 & x+1 \\ 1 & 0 & 7 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ x+1 & x & 0 & 1 & x+1 \end{vmatrix}.$$

Determine for which $x \in \mathbb{R}$ the determinant is non-zero.

12. Given the following matrix equation

$$\frac{1}{5}(X^T A)^T \cdot B^{-1} = I,$$

where A, B, X are invertible 2×2 matrices and I is the Identity matrix.

1) Compute determinant $\det(X)$ assuming that

$$\det A = 2\det B;$$

2) Find X from this equation if

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}.$$

13. Given the equation

$$A^2 + A + I = 0.$$

Prove that matrix A is non-singular matrix.

Specify a simple method for finding the inverse matrix to matrix A.

14. Determine parameters α for which the matrix

$$A = \begin{pmatrix} \alpha & 4 & 1 \\ 2 & 5 & -1 \\ 0 & \alpha & 1 \end{pmatrix}$$

has no inverse matrix?

15. Find the rank of the matrix

$$\begin{pmatrix} 1 & \lambda & -1 & 2 \\ 2 & -1 & \lambda & 5 \\ 1 & 10 & -6 & 1 \end{pmatrix},$$

for different values of parameter λ.

Part 2.

16. Find the coordinates of the point M in 3-dimension if its radius- vector forms equal angles with the coordinate axes, and the length of the radius- vector equals 3.

17. Given vectors $\vec{a}(2,-3,6)$ and $\vec{b}(-1,2,-2)$ with common initial point. Determine the coordinates of the vector $\vec{c}$, which is directed along the bisector of the angle between vectors $\vec{a}$ and $\vec{b}$, if $|\vec{c}| = 3\sqrt{42}$.

18. Find the horizontal and vertical components of the vector

$$\vec{a}(2,-1,3) \text{ on } \vec{b}(4,-1,2).$$

19. Consider two non-collinear vectors $\vec{a}$ and $\vec{b}$.
Find constants α and β if the vectors

$$\vec{c} = \alpha\vec{a} + \beta\vec{b} \text{ and } \vec{d} = (\beta+1)\vec{a} + (2-\alpha)\vec{b}$$

are equal.

20. Consider three coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. Calculate the length of the vector

$$\vec{p} = \vec{a} + \vec{b} - \vec{c},$$

if

$$|\vec{a}| = 3, \ |\vec{b}| = 2, \ |\vec{c}| = 5, \ (\vec{a},\vec{b}) = (\vec{b},\vec{c}) = 60^\circ.$$

21. Find the coordinates of the vector $\vec{x}$ if it is known that

1) vector $\vec{x}$ is collinear to the vector $\vec{a}(6;-8;-7,5)$;

2) vector $\vec{x}$ forms an acute angle with the axis Oz ;

3) $|\vec{x}| = 50$.

22. Find the projection of the vector $\vec{a}(\sqrt{2};-3;-5)$ onto the axis which forms with the coordinate axes Ox and Oz angles $\alpha = 45^\circ$, $\gamma = 60^\circ$ and the acute angle with the axis Oy.

23. Find

$$\left|[(3\vec{a}+\vec{b}),(\vec{a}-3\vec{b})]\right|$$

if it is known that

$$\left|\vec{a}\right| = 3,\ \left|\vec{b}\right| = 5,\quad (\vec{a},\vec{b}) = 60^\circ.$$

24. What condition the vectors $\vec{a}$ and $\vec{b}$ satisfy so that the vectors

$$\vec{m} = 3\vec{a}+\vec{b} \text{ and } \vec{n} = \vec{a}-3\vec{b}$$

are collinear ?

25. Calculate the area of the parallelogram which diagonals are given by the vectors

$$\vec{d}_1 = 3\vec{m}+\vec{n},\vec{d}_2 = \vec{m}-5\vec{n},$$

if

$$\left|\vec{m}\right| = \left|\vec{n}\right| = 1,\ (\vec{m},\vec{n}) = 45^\circ.$$

26. Find the vector $\vec{m}$ that is orthogonal to the axis Oz and to the vector

$$\vec{p}(8;-15;3),$$

and forms the acute angle with the axis Ox,
if

$$\left|\vec{m}\right| = 51.$$

27. Define the type of the triplet $\vec{a}$, $\vec{b}$, $\vec{c}$ if

$$\vec{a} = \vec{j}-2\vec{k},\quad \vec{b} = \vec{i}+\vec{k},\quad \vec{c} = 2\vec{i}.$$

28. Vector $\vec{c}$ is orthogonal to the vectors $\vec{a}$ and $\vec{b}$, the angle between vectors $\vec{a}$ and $\vec{b}$ equals 30° ($(\vec{a},\vec{b}) = 30^{\circ}$),

$$\left|\vec{a}\right| = 6, \quad \left|\vec{b}\right| = 3, \quad \left|\vec{c}\right| = 3.$$

Find the triple scalar product $(\vec{a},\vec{b},\vec{c})$.

29. Prove that the points

$$A(1;2;-1),\ B(0,1,5),\ C(-1,2,1),\ D(2,1,3)$$

lie in the same plane.

30. Given the matrix A

$$\mathrm{A} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

and the vector $\vec{x}$

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

1) Calculate all possible dot products between the vectors $\vec{x}$ and $\mathrm{A}\vec{x}$.
2) Compute the lengths of the vectors $\vec{x}$ and $\mathrm{A}\vec{x}$.
3) Find the angle between the vectors $\mathrm{A}\vec{x}$ and $\vec{x}$.
4) Draw a picture of these vectors in the plane.

31. Prove the identity

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}).$$

Part 3.

32. Draw the line

$$\left|z-2\right| = \left|1-2\overline{z}\right|$$

on the Complex plane.

33. Which line on the plane xOy is determined by the equation

$$z\overline{z} + i(z-\overline{z}) - 2 = 0\,?$$

34. Solve the equation

$$z^4 - 1 + i = 0\,,$$

if it is known that

$$\operatorname{Re} z < 0,\ \operatorname{Im} z > 0\,.$$

35. Calculate

$$5(\cos 10^{\circ} + i\sin 10^{\circ}) \cdot 2(\cos 80^{\circ} + i\sin 80^{\circ})\,.$$

36. Find the modulus of the complex number

$$z = \frac{x^2 - y^2 + i2xy}{xy\sqrt{2} + i\sqrt{x^4 + y^4}}\,.$$

37. Find the modulus $\left|z\right|$ and the argument $\arg z$ of the complex number

$$z = -3e^{\frac{\pi}{3}i}\,.$$

38. Find the angle by which you need to rotate the vector

$$3\sqrt{2} + i\sqrt{2}$$

to get the vector

$$-5 + i?$$

39. Represent the complex number

$$1-\cos\alpha+i\sin\alpha\,,\ 0\le\alpha\le\pi\,,$$

in the Polar form.

40. Solve the equation

$$\left|z\right|-z=1+2i\,.$$

41. Calculate

$$z^{2020}+\frac{1}{z^{2020}}\,,$$

if it is known that

$$z+\frac{1}{z}=1\,.$$

42. Given

$$(1+i)^n=(1-i)^n\,.$$

Find the integer number n.

43. Which of the following equations are the equations of the axis Ox:

$$z=0\,,$$

$$z+\overline{z}=0\,,$$

$$z=\overline{z}\,,$$

$$\arg z=0\,,$$

$$\operatorname{Im} z=0\,,$$

$$\left|z-i\right|=\left|z+i\right|,$$

$$\arg z=\frac{\pi}{2}\,,$$

$$\operatorname{Re} z=0\,,$$

$$\left|z-1\right|=\left|z+1\right|?$$

44. Reduce the product of imaginary units

$$e_9 \cdot e_2 \cdot e_4 \cdot e_3 \cdot e_5 \cdot e_7 \cdot e_8 \cdot e_1 \cdot e_6$$

to the standard form.

45. Compute the determinant

$$\begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{vmatrix}$$

by polynomial method.

Check the result by another known method.