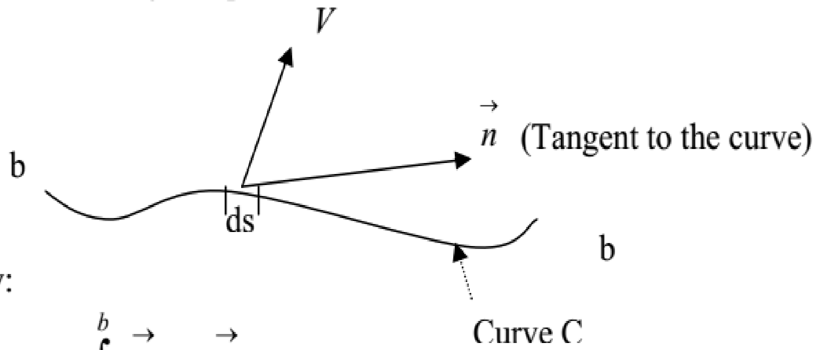
Line Integral:

Consider a vector field $\vec{v}$

$$\vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k}$$

assume that we have a curve connecting two points

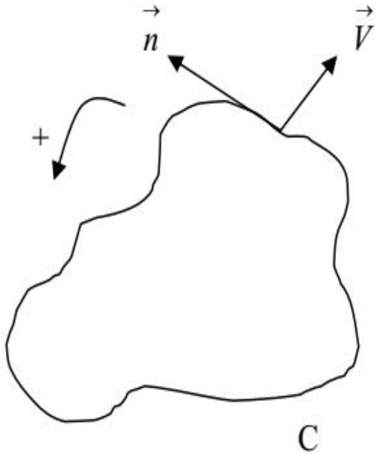


The line integral is given by:

$$\int_a^b \vec{v} \bullet \vec{n}\, ds$$

If $\subset$ is a closed curve, the line integral is

$$\oint_c \vec{v} \bullet \vec{n}\, ds$$

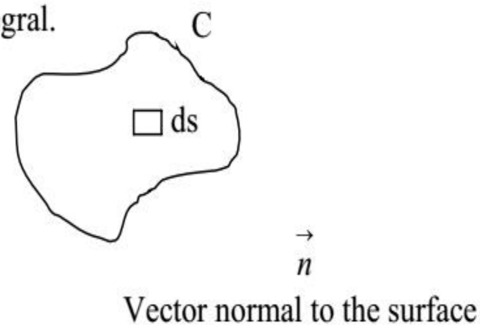


Integral is positive if counterclockwise.

Surface integral:

Consider a closed curve $\subset$ we want to compute the surface integral.

$$d\vec{S} = \vec{n}.dS$$



The surface integral is defined as:

- For a scalar field: $\iint_S p d\vec{S}$ = surface integral of a scalar p over the open surface S (results is a vector).
- For a vector field: $\iint_S \vec{V} \bullet d\vec{S}$ = surface integral of a vector $\vec{V}$ over the open surface S (result is a scalar).
- For a vector field: $\iint_S \vec{V} x d\vec{S}$ = surface integral of a vector over the open surface S (results is a vector).

If the surface is closed then:

$$\oiint_S pdS \qquad \oiint_S \vec{V} \bullet dS \qquad \iint_S \vec{V} x d\vec{S}$$

Volume Integral:

Consider a volume $\forall$, the volume integral of a scalar quantity is like the density is given by:

$\iiint_\forall \rho d\forall$ = volume integral of a scalar ρ over the volume (result is a scalar)

For a vector field:

$\iiint_\forall \vec{V} \bullet d\forall$ = volume integral of a vector $\vec{V}$ over the volume (result is a scalar)

Relation between Line, Surface and Volume Integral:

The line integral is related to the surface integral by *Stokes' theory*

$$\oint \vec{V} \bullet d\vec{S} = \iint_S \overset{\rightarrow}{\left(\Delta x \vec{V} \right)} d\vec{S}.$$

The surface integral is related to the volume integral through *Divergence Theory*:

$$\oiint_S \vec{V} \bullet d\vec{S} = \oiiint_\forall (\Delta \bullet \vec{V}) d\forall$$

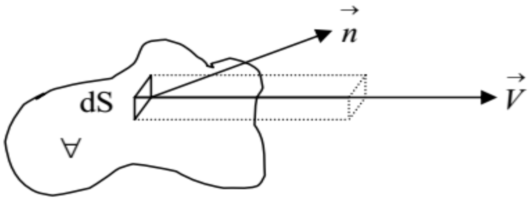
The surface integral of a scalar field like p is related to the volume integral by using *Gradient theorem:*

$$\oiint_S p \bullet d\vec{S} = \oiiint_\forall (\Delta \bullet \rho) d\forall$$

we just stated that the divergence is the time rate of change of the volume of a moving element per unit volume.

Proof:

Consider a control volume that is moving with the flow. Since this volume is moving with the fluid, it carries the same number of fluid particles. Therefore, its mass is fixed but its volume and surface are changing. Now consider as elemental surface dS moving at velocity $\vec{V}$.



The change in control volume is

$$\Delta\forall = \left[\left(\vec{V} \bullet \vec{n} \right) \bullet \Delta t \right] \overrightarrow{dS} = (\vec{v}\, \Delta t) \bullet \vec{n}\, d\vec{S} = (\vec{v}\, \Delta t) \bullet d\vec{S}$$

then the total change is the sum over all the elemental surface.

$$\oiint_S (\vec{v}\, \Delta t) \bullet d\vec{S}$$

To see how the volume is changing with time, we divide by Δt

$$\frac{D\forall}{Dt} = \lim_{t_2 \to t_1} \frac{\forall_2 - \forall_1}{t_2 - t_1} = \frac{1}{\Delta t} \oiint_S (\vec{v}\, \Delta t) \bullet d\vec{S}$$

$$\Rightarrow \frac{D\forall}{Dt} = \oiint_S \vec{V}\, d\vec{S}$$

But using the divergence theorem

$$\oiint_S \vec{V} \bullet d\vec{S} = \oiiint_\forall (\Delta \bullet \vec{V}) d\forall$$

$$\Rightarrow \frac{D\forall}{Dt} = \oiiint_\forall (\Delta \bullet \vec{V}) d\forall$$

assume that $\delta\forall$ is small enough that $\Delta \bullet \vec{V}$ =constant

$$\Rightarrow \frac{D(\delta\forall)}{Dt} = (\Delta \bullet \vec{V})d\forall$$

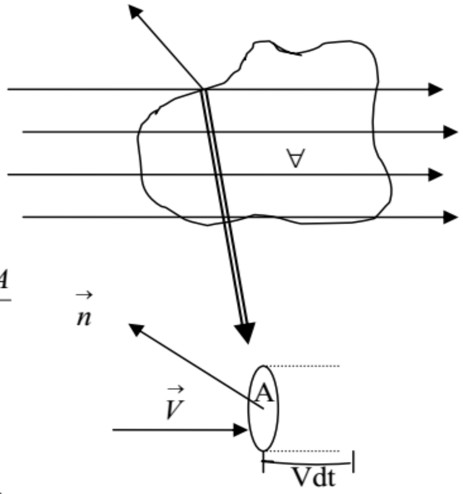
$\Rightarrow$ The divergence is: $\Delta \bullet \vec{V} = \frac{1}{\delta\forall}\frac{D\forall}{Dt}$

Which is the time rate of change of the volume of the moving element per unit volume.

Continuity Equation:

Consider a control volume $\forall$ that is fixed in the space. Unlike the derivation that we did last time where the control volume moves with the fluid that its mass is fixed but its volume and surface changing.

Since this volume we are considering is fixed, its mass is changing and its volume and surface are fixed.



$$\vec{V} = V_x \vec{i} + V_y \vec{j} + V_z \vec{k}$$

$$\text{volume} = (\vec{V} \bullet n)dtA = V_n dtA$$

mass=density* volume= $\rho V_n dtA$

Since the mass inside the volume is changing:

$\Rightarrow \frac{dm}{dt}$ =change in mass with time = mass rate= $\dot{m} = \frac{\rho V_n dtA}{dt}$

$\Rightarrow \dot{m} = \rho V_n A$

Where $V_n = \vec{V} \bullet \vec{n}$ scalar

Which means that the mass rate is equal to:

Density*the velocity component normal to the area*area

Then the mass flux:

Mass flux=mass per unit area= $\frac{\dot{m}}{A} = \rho V_n$

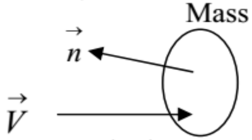
Units of $\dot{m}$ is kg/s and mass flux=kg/m^2.s

Sign convention:

mass in is negative

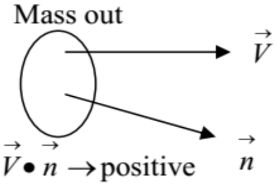
mass out is positive

why?



$\vec{V} \bullet \vec{n} \rightarrow$ Negative

$\vec{V} \bullet \vec{n} = |V||n|\cos\theta$



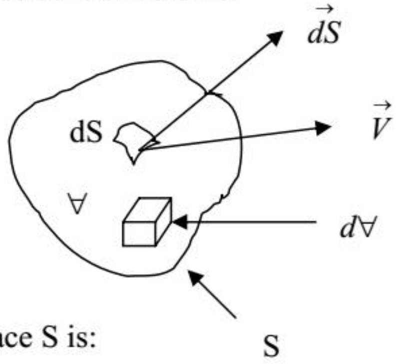
$\vec{V} \bullet \vec{n} \rightarrow$ positive

$\vec{V} \bullet \vec{n} = |V||n|\cos\theta$

Principle: mass can be neither created nor destroyed.

Let's apply this principle to the control volume.

Net mass flow out of control volume through a surface S = Time rate of decrease of mass inside control volume. (1)



Consider the left hand side:
Mass flow out of the control volume through elemental surface S is:

$$\rho(\vec{V} \bullet \vec{n})dS$$

The total (net) mass flow out of the control volume through surface S is:

$$\oiint_S \rho(\vec{V} \bullet \vec{n})d\vec{S} = \rho\vec{V} \bullet d\vec{S}$$

where $d\vec{S} = \vec{n}\, dS$

Now, for the right hand side:
Consider an elemental volume $d\forall$, the mass contained in this elemental volume is

$$\rho d\forall$$

The net mass contained inside the control volume is

$$\oiiint_\forall \rho d\forall$$

The time rate of change of mass inside the control volume is

$$\pm\frac{\partial}{\partial t}\oiiint_\forall \rho d\forall$$

\+ Increase

\- decrease

Coming back to eq.1:

$$\Rightarrow \oiint_S \rho\vec{V} \bullet d\vec{S} = -\frac{\partial}{\partial t}\oiiint_\forall \rho d\forall$$

or

$$\frac{\partial}{\partial t}\oiiint_\forall \rho d\forall + \oiint_S \rho\vec{V} \bullet d\vec{S} = 0$$

Using the divergence theorem, the second term can be written as:

$$\oiint_S \rho\vec{V} \bullet d\vec{S} = \oiiint_\forall \Delta(\rho\vec{V})d\forall$$

$$\Rightarrow \frac{\partial}{\partial t}\oiiint_{\forall} \rho d\forall + \oiiint_{\forall} \Delta(\rho \vec{V}) d\forall$$

$$\Rightarrow \oiiint_{\forall} \left[\frac{\partial}{\partial t}(\rho) + \Delta - (\rho \vec{V})\right] d\forall = 0$$

either $d\forall$ is zero(trivial solution) or

$$\frac{\partial}{\partial t}(\rho) + \Delta - (\rho \vec{V}) = 0$$

"*continuity equation is differential form*".

If the flow is steady (doesn't change with time) $\Rightarrow \frac{\partial \rho}{\partial t} = 0$

$$\Rightarrow \Delta.(\rho \vec{V}) = 0$$