Momentum equation

Let's start by writing the second law of motion:

$$\vec{F} = m\vec{a}$$

where:

$\vec{F}$: force exerted on a body
m : mass
$\vec{a}$: acceleration

or $\quad \vec{F} = \frac{d}{dt}(m\vec{V}) \qquad \vec{a} = \frac{d}{dt}\vec{V} \qquad$ ....(2)

Which means that Force= time rate of change of momentum.

Lets' consider the left hand side: *Force*.
There are tow types of force exerted on a body or fluid:

1. Body forces: gravity, electromagnetic.
2. Surface forces: pressure and shear forces.

Force= body forces + Pressure forces + Viscous forces

Lets' consider our previous control volume. The body force acting on the elemental volume $d\forall$ is

$\rho\vec{f}d\forall$

where f: is the net body force per unit mass.

The total body force is

$$\iiint_{\forall} \rho\vec{f}d\forall$$

For the surface forces, let us consider the pressure acting on the elemental surface dS

the pressure acting on dS $= -pd\vec{s}$

the net pressure force $= -\oiint_{S} pd\vec{s}$

Let's denote the viscous forces by $F_{viscous}$

The left hand side of Eq.(2):

the time rate of change of the $\frac{d}{dt}(m\vec{V})$ momentum = the net flow of momentum out of control volume $\forall$ across the S + Time rate of change of momentum due to unsteady fluctuation of flow properties inside control volume

$$\frac{d}{dt}(m\vec{V}) = \vec{G} + \vec{H}$$

Let's consider G, the flow has momentum when it enters the control volume and a different momentum when it leaves the volume.

The net momentum out= momentum out- momentum in
$\Rightarrow$ the flow of momentum is

$$(\rho\vec{V}d\vec{S})\vec{V}$$

$(\rho\vec{V}d\vec{S})\vec{V}$: the net mass flow out of control volume through S

and the net flow of momentum is

$$\vec{G} = \oiint_S (\rho\vec{V}d\vec{S})\vec{V}$$

(+ve)… mass in
(-ve)… mass out
$\vec{G}$: (+ve) momentum flow out of the control volume.
(- ve) momentum flow into of the control volume.

Now, let's consider $\vec{H}$:
The momentum of the fluid inside an elemental volume is

$$(\rho d\forall)\vec{V}$$

and for the whole control volume:

$$\oiiint_\forall \rho\vec{V}d\forall$$

Recall, H= time rate of change of momentum due to unsteady fluctuation of flow properties inside control volume.

$$\vec{H} = \frac{\partial}{\partial t}\oiiint_\forall \rho\vec{V}d\forall$$

$$\Rightarrow \frac{d}{dt}(mV) = \oiint_S (\rho\vec{V}d\vec{S})\vec{V} + \frac{\partial}{\partial t}\oiiint_\forall \rho\vec{V}d\forall$$

$$\Rightarrow \vec{F} = \frac{d}{dt}(m\vec{V})$$

$$\oiiint_\forall \rho\vec{f}d\forall - \oiint_S pd\vec{s} + \vec{F}_{viscous} = \oiint_S (\rho\vec{V}d\vec{S})\vec{V} + \frac{\partial}{\partial t}\oiiint_\forall \rho\vec{V}d\forall$$

$$\oiiint_\forall \left[\rho f_x - \frac{\partial p}{\partial x} + (f_x)_{viscous} - \nabla.(\rho u\vec{V})d\forall - \frac{\partial}{\partial t}(\rho u)\right]d\forall = 0$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho u) + \nabla.(\rho u\vec{V})d\forall = -\frac{\partial p}{\partial x} + \rho f_x + (f_x)_{viscous}$$

Similarly, the y-comp.

$$\Rightarrow \frac{\partial}{\partial t}(\rho v) + \nabla.(\rho v\vec{V})d\forall = -\frac{\partial p}{\partial y} + \rho f_y + (f_y)_{viscous}$$

and the z-comp.

$$\Rightarrow \frac{\partial}{\partial t}(\rho\omega) + \nabla.(\rho\omega\vec{V})d\forall = -\frac{\partial p}{\partial z} + \rho f_z + (f_z)_{viscous}$$

These are the momentum eqs. These eqs apply to 3-D steady, unsteady, compressible or incomp., viscous or inviscid flow.

If the flow is steady $\Rightarrow \frac{\partial}{\partial t}(.) = 0$

If the flow is inviscid $\Rightarrow F_{viscous} = 0$

And If there is no body force $\Rightarrow \vec{f} = 0$

$$\Rightarrow \qquad \nabla.(\rho u\vec{V})d\forall = -\frac{\partial p}{\partial x}$$

$$\nabla.(\rho v\vec{V})d\forall = -\frac{\partial p}{\partial y}$$

$$\nabla.(\rho\omega\vec{V})d\forall = -\frac{\partial p}{\partial z}$$