An application of the momentum equation:

Drag of a two-Dimensional Body

In the 1930s and 1940s, NASA designed some airfoil shapes and want to measure the lift and drag of these airfoils. These airfoils were placed on the wind tunnel and were made two-dimensional!! (flow around the airfoil is mainly two dimensional) by making airfoil span the entire test section.

The way they measure the lift and drag of these airfoils is by measuring the flow velocity for before the airfoil and far behind it and then use the momentum eq to compute the lift and drag.
Let's draw a control volume around this airfoil. Assume that the depth of the control volume is l.

The momentum eq :

$$-\oiint_S p\,d\vec{s} + \oiiint_\forall \rho\vec{f}\,d\forall + \vec{F}_{viscous} = \frac{\partial}{\partial t}\oiiint_\forall \rho\vec{V}\,d\forall + \oiint_S (\rho\vec{V}d\vec{S})\vec{V}$$

(1) (2) (3) (4) (5)

(1)= $-\oiint_S p\,d\vec{s} = -\iint_{abhi} p\,d\vec{s} - \iint_{def} p\,d\vec{s} - \iint_{cd} p\,d\vec{s} - \iint_{fg} p\,d\vec{s}$

but $p_{cd} = -p_{fg}$ (equal and opposite)

(2) $\oiiint_\forall \rho\vec{f}\,d\forall = 0$ (no body force)

(3)= $\vec{F}_{viscous} = (\vec{F}_{vis})_{abhi} + (\vec{F}_{vis})_{def} + (\vec{F}_{vis})_{cd} + (\vec{F}_{vis})_{fg}$

but abhi are far from the body $\Rightarrow (\vec{F}_{vis})_{abhi}=0$

$$(4) = \frac{\partial}{\partial t}\oiiint_{\forall}\rho\vec{V}d\forall = \left\{\frac{\neq 0(unsteady)}{=0 steady}\right\}$$

in this case the flow is steady $\Rightarrow \frac{\partial}{\partial t}\oiiint_{\forall}\rho\vec{V}d\forall = 0$

$$(5)\ \oiint_{S}(\rho\vec{V}d\vec{S})\vec{V} \Rightarrow$$

$$-\iint_{abhi}pd\vec{s} - \iint_{def}pd\vec{s} + (\vec{F}_{vis})_{def} = \oiint_{S}(\rho\vec{V}d\vec{S})\vec{V}$$

$$-\iint_{abhi}pd\vec{s} - R' = \oiint_{S}(\rho\vec{V}d\vec{S})\vec{V}$$

since $\quad R' = D\vec{i} + L\vec{j} + A\vec{k}$

drag lift axial

Let's consider the x-comp and this equation:

$$-\iint_{abhi}(pd\vec{s})_x - D' = \oiint_{S}(\rho\vec{V}d\vec{S})u$$

or $\quad D' = -\oiint_{S}(\rho\vec{V}d\vec{S})u - \iint_{abhi}(pd\vec{s})_x$

but since p=constant along abhi= p_∞

$$\Rightarrow \iint_{abhi}(pd\vec{s})_x = 0$$

$$\Rightarrow D' = -\oiint_{S}(\rho\vec{V}d\vec{S})u$$

$$D' = -\oiint_{a}(\rho\vec{V}d\vec{S})u - \oiint_{ab}(\rho\vec{V}d\vec{S})u - \oiint_{bh}(\rho\vec{V}d\vec{S})u - \oiint_{Ri}(\rho\vec{V}d\vec{S})u - \oiint_{cd}(\rho\vec{V}d\vec{S})u - \oiint_{de}(\rho\vec{V}d\vec{S})u - \oiint_{ef}(\rho\vec{V}d\vec{S})u - \oiint_{fg}(\rho\vec{V}d\vec{S})u$$

$$D' = -\oiint_{a}(\rho\vec{V}d\vec{S})u - \oiint_{bh}(\rho\vec{V}d\vec{S})u$$

but at ai (station 1) and bh (station 20 boundaries $d\vec{S} = (dy)1$

$$D' = -\{\int_{i}^{a}(\rho_1 u_1 dy)u_1 + \int_{hi}^{b}(\rho_2 u_2 dy)u_2\}$$

$$D' = -\int_{i}^{a}(\rho_1 u_1^2 dy) - \int_{hi}^{b}(\rho_2 u_2^2 dy)$$

but continuity eq $\qquad \oiint_{S}\rho\vec{V}d\vec{S} = 0$ (steady flow)

$$-\int_{i}^{a}(\rho_1 u_1^2 dy) + \int_{h}^{b}(\rho_2 u_2^2 dy) = 0$$

Multiply by u_1:

$$\int_{hi}^{b} (\rho_2 u_1 u_2 dy) = \int_{i}^{a} (\rho_1 u_1^2 dy)$$

$$D' = \int_{hi}^{b} (\rho_2 u_1 u_2 dy) - \int_{h}^{b} (\rho_2 u_2^2 dy)$$

$$D' = \int_{h}^{b} (\rho_2 u_2 (u_1 - u_2) dy)$$

if the flow is incompressible $\rho_2 = \rho_1$

$$D' = \rho \int_{h}^{b} (u_2 (u_1 - u_2) dy)$$

Energy Equation:

Physical principle: energy can be neither created nor destroyed; it can only change in form.

First law of thermodynamics $\delta q + \delta \omega = de$

δq : heat added to the system

$\delta \omega$: work done on the system by the surroundings.

δe : the change of internal energy

δ is path differential

d is an exact differential

$$\int_{1}^{2} \delta q \neq Q_2 - Q_1$$

let's apply this law to the fluid inside the control volume $\forall$

$B_1 + B_2 = B_3$

B_1 : rate of heat added to fluid inside $\forall$

B_2 : rate of work done on fluid inside $\forall$

B_3 : rate of change of energy of fluid as it flows through $\forall$

Let's consider the first term B_1 :

Assume that the control volume $\forall$ is subjected to heat addition. Let's denoted the heat addition per unit mass as $\dot{q}$ (J/S.kg)

Then, the rate of volumetric heating $\iiint_{\forall} \dot{q} \rho d\forall$

Let's also denote the rate of heat addition due to viscous effects by $\dot{Q}_{viscous}$

For B_2 :

rate of work done on moving body= $\vec{F}.\vec{V}$

work= $\vec{F}.d\vec{r}$

time rate of doing work= $\vec{F}.\frac{d\vec{r}}{dt}=\vec{F}.\vec{V}$

last time we talked about the types of force

$$\vec{F}=\vec{F}_{body}+\vec{F}_{pressure}+\vec{F}_{viscous}$$

$$\vec{F}_{body}=\oiiint_{\forall}\rho\vec{f}d\forall$$

$$\vec{F}_{pressure}=-\oiint_{S}pd\vec{s}$$

rate of work done on $\forall=\vec{F}.\vec{V}$

$$B_2=\oiiint_{\forall}\rho\vec{f}d\forall-\oiint_{S}pd\vec{s}+\dot{W}_{viscous}$$

For B_3:

The rate of change of energy of fluid as it flows through $\forall$

The flow enters the control volume with certain mass and energy and leave with a different mass and energy.

Net rate of flow of total energy across the control surface=

$$\oiint_{S}(\rho\vec{V}d\vec{S})\vec{V}.(e+\frac{V^2}{2})$$

Also inside the control volume, there is a change of energy due to unsteady flow

$$\oiiint_{\forall}(\rho\vec{V}d\vec{S})\vec{V}.(e+\frac{V^2}{2})$$

time rate of change of total energy inside $\forall$ due to unsteady flow variables=

$$\frac{\partial}{\partial t}\oiiint_{\forall}\rho.(e+\frac{V^2}{2})d\vec{V}$$

$$\Rightarrow B_3=\frac{\partial}{\partial t}\oiiint_{\forall}\rho.(e+\frac{V^2}{2})d\vec{V}+\oiiint_{\forall}(\rho\vec{V}d\vec{S})\vec{V}.(e+\frac{V^2}{2})$$

Recall $B_1+B_2=B_3$

$$\oiiint_{\forall}\dot{q}\rho d\forall+\dot{Q}_{viscous}+\oiiint_{\forall}\rho\vec{f}d\forall-\oiint_{S}(pd\vec{s})\vec{V}+\dot{W}_{viscous}$$

$$=\frac{\partial}{\partial t}\oiiint_{\forall}\rho.(e+\frac{V^2}{2})d\vec{V}+\oiint_{S}(\rho\vec{V}d\vec{S}).(e+\frac{V^2}{2})$$

Using the divergence theorem:

$$\oiint_{S}(pd\vec{S})\vec{V}=\oiint_{S}(p\vec{V})d\vec{S}=\oiiint_{\forall}\nabla.(p\vec{V})d\forall$$

$$\oiiint_{\forall}(\rho.(e+\frac{V^2}{2})\vec{V}d\vec{S}=\oiiint_{\forall}\nabla.(e+\frac{V^2}{2})\vec{V}d\forall$$

$$\rho\dot{q}+\dot{Q}_{viscous}+\rho(\vec{f}.\vec{V})-\nabla.(p\vec{V})+\dot{W}_{viscous}=\frac{\partial}{\partial t}\left[\rho.(e+\frac{V^2}{2})\right]+\nabla\left[\rho.(e+\frac{V^2}{2})\right]$$

Flow is steady $\Rightarrow\ \frac{\partial}{\partial t}=0$

inviscid $\Rightarrow \dot{W}_{viscous} = 0 = \dot{Q}_{viscous}$
no heat addition $\Rightarrow \dot{q} = 0$
no body force $\Rightarrow$ f=0

$$\Rightarrow \nabla\left[\rho.(e + \frac{V^2}{2})\right] = -\nabla(\rho.\vec{V})$$

Substantial Derivative:

$$\rho_1 = \rho(x_1, y_1, z_1, t_1)$$

$$\rho_2 = \rho(x_2, y_2, z_2, t_2)$$

$$\rho_2 = \rho_1 + \left(\frac{\partial p}{\partial x}\right)(x_2 - x_1) + \left(\frac{\partial p}{\partial y}\right)(y_2 - y_1) + \left(\frac{\partial p}{\partial z}\right)(z_2 - z_1) + \left(\frac{\partial p}{\partial t}\right)(t_2 - t_1) + H.O.T$$

Ignore HOT and divide by $t_2 - t_1$

$$\frac{\rho_2 - \rho_1}{t_2 - t_1} = \left(\frac{\partial p}{\partial t}\right) + \frac{\rho_1}{(t_2 - t_1)} + \left(\frac{\partial p}{\partial x}\right)\left(\frac{x_2 - x_1}{t_2 - t_1}\right) + \left(\frac{\partial p}{\partial y}\right)\left(\frac{y_2 - y_1}{t_2 - t_1}\right) + \left(\frac{\partial p}{\partial z}\right)\left(\frac{z_2 - z_1}{t_2 - t_1}\right)$$

$\frac{\rho_2 - \rho_1}{t_2 - t_1}$ is the average time rate of change in the density as the fluid moves from point 1 to 2.

$$\lim_{t_2 - t_1} \frac{\rho_2 - \rho_1}{t_2 - t_1} = \frac{D\rho}{Dt}$$ (Instantaneous time rate of change)

Similarly:

$$\lim_{t_2 - t_1} \frac{x_2 - x_1}{t_2 - t_1} = u$$

$$\lim_{t_2 - t_1} \frac{y_2 - y_1}{t_2 - t_1} = v$$

$$\lim_{t_2 - t_1} \frac{z_2 - z_1}{t_2 - t_1} = w$$

$$\frac{D\rho}{Dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}$$

or:

$$\frac{D(\,)}{Dt} = \frac{\partial(\,)}{\partial t} + u\frac{\partial(\,)}{\partial x} + v\frac{\partial(\,)}{\partial y} + w\frac{\partial(\,)}{\partial z}$$

but:

$$\vec{V} = u\,\vec{i} + v\,\vec{j} + w\,\vec{k}$$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\frac{D}{Dt} = \underbrace{\frac{\partial}{\partial t}}_{\substack{\text{local}\\ \text{derivative}}} + \underbrace{\vec{V}.\nabla(\,)}_{\substack{\text{convective}\\ \text{derivative}}}$$