COMBINATORICS

Course Overview:

This is an important branch of discrete mathematics concerns properties of collections of subsets of a finite set. There are many beautiful and fundamental results, and there are still many basic open questions. The aim of the course is to introduce this very active area of mathematics, with many connections to other fields.

Learning Outcomes:

The student will have developed an appreciation of the combinatorics of finite sets.

Course Synopsis:

Chains and antichains. Sperner's Lemma. LYM inequality. Dilworth's Theorem.

Shadows. Kruskal-Katona Theorem.

Intersecting families. Erdos-Ko-Rado Theorem. Cross-intersecting families.

VC-dimension. Sauer-Shelah Theorem.

t-intersecting families. Fisher's Inequality. Frankl-Wilson Theorem. Application to Borsuk's Conjecture.

Combinatorial Nullstellensatz.

Please refer to the following books as your reading list

1. Bela Bollobás, *Combinatorics*, CUP, 1986.
2. Stasys Jukna, *Extremal Combinatorics*, Springer, 2007

Notation and basic definitions

For a set X, we write $\mathcal{P}(X) = \{A : A \subseteq X\}$ for its power set (i.e. the set of all its subsets). A *family of sets $\mathcal{F}$ on a ground set X* is a subset $\mathcal{F} \subseteq \mathcal{P}(X)$. A family of sets is also called a *set system* or a *hypergraph*. For instance, with ground set $X = \{1, 2, 3, 4\}$, we could have the hypergraph $\mathcal{F} = \{1, 13, 124, 34\}$: note that we have dropped some brackets and commas for convenience (strictly speaking we should write $\mathcal{F} = \{\{1\}, \{1, 3\}, \{1, 2, 4\}, \{3, 4\}\}$, but this is more cumbersome).

Most of this course concerns properties of set systems with a finite ground set. So from now on *all sets will be assumed to be finite unless stated otherwise.*

For $n \geq 1$, we write $[n] := \{1, 2, \ldots, n\}$. We will usually write $\mathcal{P}(n)$ instead of $\mathcal{P}([n])$. Note that $|\mathcal{P}(n)| = 2^n$.

We will refer to sets of size k as *k-sets*. For $k \geq 0$ we write $X^{(k)} = \{A \in \mathcal{P}(X) : |A| = k\}$ for the set of all subsets of X of size k. We define $X^{(<k)}, X^{(\leq k)}, X^{(>k)}$ and $X^{(\geq k)}$ in the obvious way. A family $\mathcal{F} \subseteq \mathcal{P}(X)$ is *k-uniform* if $\mathcal{F} \subseteq X^{(k)}$.

We think of $X^{(0)}, X^{(1)}, \ldots$ as the *layers* of $\mathcal{P}(X)$, and refer to $X^{(i)}$ as the *ith layer*. So the ith layer of $\mathcal{P}(n)$ is $[n]^{(i)}$, which has cardinality $\binom{n}{i}$. The smallest layers of $\mathcal{P}(n)$ are the 0th layer and the nth layer: these are $\{\emptyset\}$ and $\{[n]\}$ respectively, and both have size 1. If n is even, the largest layer is $[n]^{(n/2)}$; if n is odd there are two largest layers, $[n]^{(\lfloor n/2 \rfloor)}$ and $[n]^{(\lceil n/2 \rceil)}$.

There are many ways of looking at the power set $\mathcal{P}(n)$. For instance:

- We can think of $\mathcal{P}(n)$ as a collection of sets with a partial order given by set containment: $A \subseteq B$. We will talk more about this later.

- We can turn $\mathcal{P}(n)$ into a graph: the *discrete cube* Q_n is the graph with vertex set $\mathcal{P}(n)$ and an edge between A and B if and only if $|A \triangle B| = 1$. Here $A \triangle B$ is the *symmetric difference*: $A \triangle B := (A \setminus B) \cup (B \setminus A)$.

- We can turn $\mathcal{P}(n)$ into an abelian group or a vector space $\mathbb{F}_2^n$ by identifying each $A \subseteq [n]$ with its *characteristic vector*

$$\chi_A = (\chi_A(1), \ldots, \chi_A(n)) \in \{0,1\}^n,$$

where

$$\chi_A(i) := \begin{cases} 1 & i \in A \\ 0 & i \notin A \end{cases}$$

We will move back and forward between these perspectives throughout the course.