Chains, antichains and shadows

2.1 Sperner's Lemma, LYM Inequality and Dilworth's Theorem

A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is a *chain* if, for all $A, B \in \mathcal{A}$, either $A \subseteq B$ or $B \subseteq A$. A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is an *antichain* if, for all distinct $A, B \in \mathcal{A}$, we have $A \not\subseteq B$ and $B \not\subseteq A$.

For instance, $\{\emptyset, 1, 1234\}$ is a chain in $\mathcal{P}(4)$, and $\{12, 234, 14\}$ is an antichain. For any set X, the layers $X^{(0)}, X^{(1)}, \ldots$ are antichains.

How large can a chain in $\mathcal{P}(n)$ be? This has a simple answer.

Proposition 1. *Every chain in $\mathcal{P}(n)$ has at most $n+1$ elements. There are $n!$ different maximal chains with $n+1$ elements, and every chain C is contained in some chain of size $n+1$.*

Proof. Exercise. ∎

How large can an antichain be? This is more interesting. We noted above that the layers of $\mathcal{P}(n)$ are antichains: the largest of these has size $\binom{n}{\lfloor n/2 \rfloor}$.

Theorem 2. (Sperner's Lemma) *An antichain in $\mathcal{P}(n)$ has size at most $\binom{n}{\lfloor n/2 \rfloor}$.*

We shall see two proofs of Sperner's Lemma.

For the first proof, we will need a result from Graph Theory. Let us recall that a graph $G = (V, E)$ is bipartite with vertex classes X, Y if $X \cup Y$ is a partition of V such that every edge of G contains one vertex from each of X

and Y. The *neighbourhood* $\Gamma(v)$ of a vertex $v \in V$ is $\Gamma(v) = \{u : vu \in E(G)\}$, and for $S \subseteq V$ we write $\Gamma(S) = \cup_{v \in X}\Gamma(v)$. A *complete matching from X to Y* is a collection of vertex-disjoint edges such that every vertex in X is incident to some edge in M. Here is the result that we will need.

Theorem 3. (Hall's Theorem) *Let $G = (V, E)$ be a bipartite graph with vertex classes X and Y. Then G has a complete matching from X to Y if and only if, for all $S \subseteq X$, we have*

$$|\Gamma(S)| \geq |S|. \tag{2.1}$$

We will refer to equation (2.1) as *Hall's Condition*. Note that G has a complete matching then (2.1) is clearly necessary. The point of Hall's Theorem is that it is also *sufficient*.

Sperner's Lemma will follow quickly from the following result.

Lemma 4. *There is a partition of $\mathcal{P}(n)$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains.*

Proof. Consider the subgraph of the discrete cube Q_n between the vertices in two consecutive layers. We claim that:

1. for $r < n/2$, there is a complete matching from $[n]^{(r)}$ to $[n]^{(r+1)}$;
2. for $r > n/2$, there is a complete matching from $[n]^{(r)}$ to $[n]^{(r-1)}$.

If we glue these matchings together, we obtain a collection of $\binom{n}{\lfloor n/2 \rfloor}$ chains that partition $\mathcal{P}(n)$.

It is therefore enough to prove that these matchings exist. For $r < n/2$, we consider the bipartite subgraph G of Q_n induced by $[n]^{(r)} \cup [n]^{(r+1)}$. This has bipartition $([n]^{(r)}, [n]^{(r+1)})$ and an edge between $A \in [n]^{(r)}$ and $B \in [n]^{(r+1)}$ iff $A \subseteq B$. In order to prove that there is a complete matching, it is enough to verify Hall's Condition (2.1). We will do this by a double counting argument.

Consider a set $S \subseteq [n]^{(r)}$, and let $T = \Gamma(S)$. Each $A \in S$ has degree $n - r$ in G (as there are $n - r$ ways to add an element to A to get an $(r + 1)$-set). So the number of edges between S and T is

$$e(S, T) = (n - r)|S|.$$

On the other hand, each $B \in [n]^{(r+1)}$ has degree $r + 1$ (as there are $r + 1$ choices for an element to delete from B to get an r-set). So we have[1]

$$e(S, T) \leq (r + 1)|T|$$

[1] Note that we may not have equality here, as there may be edges incident with T that are not incident with S.

Putting these two bounds together, we get

$$|T| \geq \frac{n-r}{r+1}|S| \geq |S|,$$

as $r < n/2$ (and so $r \leq (n-1)/2$). So Hall's Condition is satisfied, and hence there is a complete matching from $[n]^{(r)}$ to $[n]^{(r+1)}$.

For $r > n/2$, we can argue similarly. Alternatively we can consider the effect of replacing every set by its complement. ■

Remarks:

1. Note that we cannot partition $\mathcal{P}(n)$ into fewer than $\binom{n}{\lfloor n/2 \rfloor}$ chains, because no two sets from the antichain $[n]^{(\lfloor n/2 \rfloor)}$ can belong to the same chain.

2. The chains that we get from Lemma 4 could be very 'asymmetric'. For instance, the chain that starts at $\emptyset$ could finish on a middle layer (rather than continuing all the way up to $[n]$).

3. As an exercise you should calculate: what is (roughly) the *average* length of the chains in the partition of $\mathcal{P}(n)$ given by Lemma 4?

We can now prove Sperner's Lemma.

First proof of Sperner's Lemma. This is now easy. A chain and an antichain meet in at most one element. We have partitioned $\mathcal{P}(n)$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, so no antichain can have more than $\binom{n}{\lfloor n/2 \rfloor}$ elements. ■

Sperner's Lemma tells us the maximal size of an antichain, but what can we say about uniqueness? And what happens if we start using sets of different sizes? The LYM Inequality (named after Lubell, Yamamoto and Meshalkin, who all gave independent proofs of the result) gives a much more refined picture.

Theorem 5. (LYM Inequality) *Let $\mathcal{F} \subseteq \mathcal{P}(n)$ be an antichain. Then*

$$\sum_{i=0}^{n} \frac{|\mathcal{F} \cap [n]^{(i)}|}{\binom{n}{i}} \leq 1. \tag{2.2}$$

Furthermore, we have equality in (2.2) if and only if $\mathcal{F} = [n]^{(i)}$ for some i.

We will give two proofs of the LYM Inequality. The first one will use a "local" version of the inequality.

Let $\mathcal{F} \subseteq X^{(k)}$ be a k-uniform family on X. The *(lower) shadow* $\partial\mathcal{F}$ *of* $\mathcal{F}$ is

$$\partial\mathcal{F} := \{B \in X^{(k-1)} : B \subseteq A \text{ for some } A \in \mathcal{F}\}.$$