CHAINS, ANTICHAINS AND SHADOWS

Sperner's Lemma, LYM Inequality and Dilworth's Theorem CONTINUED

Lemma 6. (Local LYM Inequality) *Let $\mathcal{F} \subseteq [n]^{(r)}$. Then*

$$\frac{|\partial\mathcal{A}|}{\binom{n}{r-1}} \geq \frac{|\mathcal{A}|}{\binom{n}{r}}.$$

We have equality if and only if $\mathcal{A} = \emptyset$ or $\mathcal{A} = [n]^{(r)}$.

Proof. Once more, we use double counting of edges in Q_n, here between $\mathcal{A}$ and $\partial\mathcal{A}$. Thus we are double counting elements of

$$E = \{(A, B) : A \in \mathcal{A}, B \in \partial\mathcal{A}, B \subseteq A\}.$$

Each element $A \in \mathcal{A}$ contains r sets of size $r - 1$, so

$$|E| = r|\mathcal{A}|.$$

Each element $B \in \partial\mathcal{A}$ is contained in $n-r+1$ sets of size r (not all of which need be in $\mathcal{A}$). So

$$|E| \leq (n - r + 1)|\partial\mathcal{A}|.$$

It follows that

$$(n - r + 1)|\partial\mathcal{A}| \geq r|\mathcal{A}|, \tag{2.3}$$

and so

$$\frac{|\mathcal{A}|}{\binom{n}{r}} \leq |\partial\mathcal{A}| \cdot \frac{n-r+1}{r} \cdot \frac{1}{\binom{n}{r}} = \frac{|\partial\mathcal{A}|}{\binom{n}{r-1}},$$

as required.

If we have equality then we must have equality in (2.3), so for every $B \in \partial\mathcal{A}$ and every $i \notin B$ we have $B \cup i \in \mathcal{A}$. If $\mathcal{A} \neq \emptyset, [n]^{(r)}$ then choose r-sets $A_1 \in \mathcal{A}$ and $A_2 \notin \mathcal{A}$ with $|A_1 \triangle A_2|$ as small as possible. We can choose $a_1 \in A_1 \backslash A_2$ and $a_2 \in A_2 \backslash A_1$: then $A_1 - a_1$ is in $\partial\mathcal{A}$ and so $A_3 := (A_1 - a_1) \cup a_2$ is in $\mathcal{A}$. But this gives a contradiction, as $|A_3 \triangle A_2| < |A_1 \triangle A_2|$. ■

Remark: In the last paragraph, what we really used was the fact that E is the edge set of a connected graph. Note also that we are being a little informal with notation in expressions like $(A_1 - a_1) \cup a_2$. As with leaving out brackets in listing set elements, this is fine as long as it is not ambiguous.

We use the Local LYM Inequality to prove the LYM Inequality.

Proof of the LYM Inequality. Let $\mathcal{F} \subseteq \mathcal{P}(n)$ be an antichain and, for $i = 0, \ldots, n$, define

$$\mathcal{F}_i := \mathcal{F} \cap [n]^{(i)}.$$

The proof proceeds by 'pushing down one layer at a time'. Define $\mathcal{G} \subseteq [n]^{(i)}$ recursively by $\mathcal{G}_n = \mathcal{F}_n$ and, for $r < n$,

$$\mathcal{G}_r := \mathcal{F}_r \cup \partial\mathcal{G}_{r+1}.$$

Note that $\mathcal{F}_r$ and $\partial\mathcal{G}_{r+1}$ are disjoint, as every set in $\partial\mathcal{G}_{r+1}$ is contained in some element of $\mathcal{F}$, and $\mathcal{F}$ is an antichain.

We claim that, for $r = 0, \ldots, n$,

$$\frac{|\mathcal{G}_r|}{\binom{n}{r}} \geq \sum_{i=r}^{n} \frac{|\mathcal{F}_i|}{\binom{n}{i}}. \tag{2.4}$$

We prove this by (downwards) induction. For $r = n$ it is immediate. Now suppose it is true for $r + 1$: we show it is true for r. We have

$$\begin{aligned}
\frac{|\mathcal{G}_r|}{\binom{n}{r}} &= \frac{|\mathcal{F}_r|}{\binom{n}{r}} + \frac{|\partial\mathcal{G}_{r+1}|}{\binom{n}{r}} && \text{as } \mathcal{F}, \partial\mathcal{G}_{r+1} \text{ disjoint} \\
&\geq \frac{|\mathcal{F}_r|}{\binom{n}{r}} + \frac{|\mathcal{G}_{r+1}|}{\binom{n}{r+1}} && \text{by Local LYM} \\
&\geq \frac{|\mathcal{F}_r|}{\binom{n}{r}} + \sum_{i=r}^{n} \frac{|\mathcal{F}_i|}{\binom{n}{i}} && \text{by induction}
\end{aligned}$$

as required. We conclude that (2.4) holds for every r.

Setting $r = 0$ in (2.4), we get

$$1 \geq \frac{|\mathcal{G}_0|}{\binom{n}{0}} \geq \sum_{i=0}^{n} \frac{|\mathcal{F}_i|}{\binom{n}{i}},$$

which gives (2.2) as required.

If we have equality at the end, we must have had equality at each application of Local LYM, and so we have $\mathcal{G}_r = \emptyset$ or $\mathcal{G}_r = [n]^{(r)}$ for each r. This implies that $\mathcal{F}_r = [n]^{(r)}$ for some r, and $\mathcal{F}_i = \emptyset$ for all other i. ■

We give a second proof of the LYM Inequality. It gives a very slick proof of the LYM inequality, but does not give the extremal set systems.

Second proof of the LYM Inequality. Let $\mathcal{F}$ be an antichain in $\mathcal{P}(n)$. Choose a maximal chain $C = (A_0, \dots, A_n)$, where $\emptyset = A_0 \subseteq \dots A_n = [n]$, uniformly at random from all $n!$ maximal chains. For $A \in \mathcal{F}$ with $|A| = k$ we have

$$\mathbb{P}[A \in C] = \frac{k!(n-k)!}{n!} = \frac{1}{\binom{n}{k}}.$$

The events $(A \in C)_{A \in \mathcal{F}}$ are pairwise disjoint, so by the union bound we have

$$1 \geq \sum_{A \in \mathcal{F}} \mathbb{P}[A \in C] = \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}},$$

which gives the LYM Inequality. ■

The LYM Inequality gives a second proof of Sperner's Lemma. In fact we get a bit more, as it allows us to characterize the extremal set systems.

Corollary 7. *Let $\mathcal{F} \subseteq \mathcal{P}(n)$ be an antichain. Then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$, with equality if and only if if $\mathcal{F} = [n]^{(\lfloor n/2 \rfloor)}$ or $\mathcal{F} = [n]^{\lceil n/2 \rceil}$.*

Proof. Let $\mathcal{F} \subseteq \mathcal{P}(n)$ be an antichain. By the LYM Inequality, we have

$$\begin{aligned} 1 &\geq \sum_{i=0}^{n} \frac{|\mathcal{F} \cap [n]^{(i)}|}{\binom{n}{i}} \\ &\geq \sum_{i=0}^{n} \frac{|\mathcal{F} \cap [n]^{(i)}|}{\binom{n}{\lfloor n/2 \rfloor}} \\ &= \frac{|\mathcal{F}|}{\binom{n}{\lfloor n/2 \rfloor}}, \end{aligned}$$

where the second inequality follows because $\binom{n}{\lfloor n/2 \rfloor}$ is the largest binomial coefficient. If we have equality in the first line (where we applied LYM) then we must have $\mathcal{F} = [n]^{(i)}$ for some i. But then to get equality in the second line we must have $i = \lfloor n/2 \rfloor$ or $i = \lceil n/2 \rceil$. ■

Lemma 4 and Sperner's Lemma together tell us that the minimum number of chains in a partition of $\mathcal{P}(n)$ is equal to the maximum size of an antichain. This is a special case of a more general theorem about partially ordered sets.

A *partially ordered set* or *poset* $(P, \leq)$ is a set P with a relation $\leq$ such that, for all $a, b, c \in P$, we have

- $a \leq a$ (reflexivity),
- if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity),
- if $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry).

We write $a < b$ to mean $a \leq b$ and $a \neq b$.

Two elements $a, b \in P$ are *comparable* if either $a \leq b$ or $b \leq a$. Two elements are *incomparable* if they are not comparable.

A set $C \subseteq P$ is a *chain* if every pair of elements from C is comparable. A set $A \subseteq P$ is an *antichain* if every pair of distinct elements from A is incomparable.

Any set system $\mathcal{F}$ becomes an antichain under the containment relation $\subseteq$. You should check that the chains and antichains in this poset are the same as in the earlier definition, and that a chain and an antichain can have at most one common element.

We say that a collection of chains *covers* a poset P if every element of P is contained in one of the chains. We can now state our main theorem about posets.