CHAINS, ANTICHAINS AND SHADOWS

Sperner's Lemma, LYM Inequality and Dilworth's Theorem CONTINUED

Theorem 8. (Dilworth's Theorem) *Let $(P, \leq)$ be a finite poset. The minimum number of chains needed to cover P is equal to the maximum size of an antichain.*

Proof. Since a chain and an antichain meet in at most one element, it is clear that the number of chains in any cover is at least as large as the maximum size of an antichain. So we need only prove that there is a cover with this many chains.

We argue by induction on $|P|$. The statement is immediate for $|P| = 0$, so we assume $|P| > 0$ and we have proved the statement for smaller posets.

Let m be the maximum size of an antichain and let C be a maximal chain (i.e. a chain C such that $C \cup \{a\}$ is not a chain, for any $a \in P \setminus C$). [Note that maximal chain does not necessarily mean chain of maximal size!]

If $P \setminus C$ contains no antichain of size m then we are done by induction: we can cover $P \setminus C$ with $m - 1$ chains, and then add C to get a cover of P with m chains.

Otherwise, there is an antichain of size m in $P \setminus C$, say $A = \{a_1, \ldots, a_m\}$. Let

$$S^+ := \{x \in P : x \geq a_i \text{ for some } a_i \in A\}$$

and

$$S^- := \{x \in P : x \leq a_i \text{ for some } a_i \in A\}.$$

Then $S^+ \cap S^- = A$, as A is an antichain. Also, $S^+ \cup S^- = P$ as A is maximal (if there were $x \notin S^+ \cup S^-$ then $A \cup \{x\}$ would be a larger antichain).

We now check that S^+ and S^- are both *proper* subsets of P. Let the maximal chain C have elements $\{c_1, \ldots, c_k\}$, where $c_1 < \cdots < c_k$. Since C is maximal, c_k is a maximal element of P and c_1 is a minimal element of P. Then:

- $c_k \notin S^-$, because c_k is a maximal element of P and $c_k \notin A$
- $c_1 \notin S^+$, because c_k is a minimal element of P and $c_1 \notin A$.

Therefore, S^+ and S^- are proper subsets of P, and so by induction we can partition S^+ into m chains $C_1^+, \ldots, C_m^+$, and we can partition S^+ into m chains $C_1^-, \ldots, C_m^-$.

Since $A \subseteq S^+$ and A meets each chain C_i^+ in at most one elemen, we see that each C_i^+ must contain exactly one element of A. Relabelling if necessary, we may assume that $a_i \in C_i^+$, for $i = 1, \ldots, m$. Similarly, we may assume that $a_i \in C_i^-$, for $i = 1, \ldots, m$.

If a_i is not maximal in C_i^-, there exists $b \in C_i^-$ with $b > a_i$. But then, since $b \in S^-$, we can find $a_j \in A$ with $b \leq a_j$. This means $a_i < b \leq a_j$, so $a_i < a_j$, which gives us a contradiction as A is an antichain. So a_i is maximal in C_i^-. Similarly, a_i is minimal in C_i^+.

Finally, we glue C_i^+ and C_i^- together (the 'gluing point' being a_i) to obtain a partition of P into m chains. ■

There is a 'dual' of Dilworth's Theorem: the minimum number of antichains in a cover P is equal to the maximum size of a chain. The proof of this is an exercise on the first example sheet.

Symmetric chains and Littlewood-Offord

If $z_1, \ldots, z_n \in \mathbb{C}$ are such that $|z_i| \geq 1$, how many of the 2^n sums $\sum\limits_{i \in I} z_i$, where $I \subseteq [n]$, can equal 0? This question was raised in 1938, by Littlewood and Offord. A few years later, Erdős found a neat solution in the real case.

Theorem 9. (Erdős) *Suppose that $x_1, \ldots, x_n \in \mathbb{R}$ satisfy $|x_i| \geq 1$ for all i. For every $\alpha \in \mathbb{R}$, there are at most $\binom{n}{\lfloor n/2 \rfloor}$ subsets $I \subseteq [n]$ such that $\sum_{i \in I} x_i \in [\alpha, \alpha + 1)$.*

Note that the bound in this theorem is sharp: take $x_i = 1$ for all i, and set $\alpha = \binom{n}{\lfloor n/2 \rfloor}$.

Proof of Theorem 9. Consider first the effect of replacing x_i with $-x_i$. Let $S_1, \ldots, S_N$ denote the $N = 2^{n-1}$ sums corresponding to subsets $I \subseteq [n]$ that do not contain i. Then the full collection of sums of subsets is

$$S_1, \ldots, S_N, S_1 + x_i, S_2 + x_i, \ldots, S_N + x_i.$$

If we replace x_i with $-x_i$, this becomes

$$S_1, \dots, S_N, S_1 - x_i, S_2 - x_i, \dots, S_N - x_i,$$

which is just a translation (and reordering) of the first collection. So replacing x_i by $-x_i$ does not affect the truth of the theorem.

We may assume that $x_i \geq 1$ for all i. But now, if we take any $\binom{n}{\lfloor n/2 \rfloor} + 1$ subsets of $[n]$, then by Sperner's Lemma there must be some pair I, J with $I \subsetneq J$. Then $\sum_{i \in J} x_i \geq \sum_{i \in I} x_i + 1$, so we cannot have both sums in $[\alpha, \alpha + 1)$. ■

We will also see a solution to the Littlewood-Offord problem in the complex case, but first let us think a bit more about chains.

A chain $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_m$ in $\mathcal{P}(n)$ is *symmetric* if $|C_{i+1}| = |C_i| + 1$, for all $i = 1, \dots, m-1$, and $|C_1| + |C_m| = n$. We saw in Proposition 4 that $\mathcal{P}(n)$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains, but the chains we obtained could be asymmetric.

Proposition 10. *For $n \geq 1$, there is a partition of $\mathcal{P}(n)$ into symmetric chains.*

Proof. We argue by induction on n. The case $n = 1$ is easy! So suppose that $n > 1$ and that $\mathcal{C}_1, \dots, \mathcal{C}_m$ is a partition of $\mathcal{P}(n-1)$ into symmetric chains. For each chain $\mathcal{C}_i$, say $\mathcal{C}_i = \{A_1, \dots A_k\}$ with $A_1 \subseteq \ldots \subseteq A_k$, we define two chains in $\mathcal{P}(n)$:

$$\mathcal{C}_i' := \{A_1, A_2, \dots, A_k, A_k \cup n\}$$

and

$$\mathcal{C}_i'' := \{A_1 \cup n, \dots, A_{k-1} \cup n\}.$$

If $k = 1$ then $\mathcal{C}_i''$ is empty: we discard these empty chains. The resulting chains give a partition of $\mathcal{P}(n)$ into symmetric chains. ■

A symmetric chain decomposition of $\mathcal{P}(n)$ has $\binom{n}{\lfloor n/2 \rfloor}$ chains, each of which contains an element from the middle layer (or contains an element from each middle layer, if n is odd). For each $i \leq n/2$, there are

$$\binom{n}{i} - \binom{n}{i-1}$$

chains of size $n - 2i + 1$, and these run from the ith layer to the $(n-i)$th layer.

We now return to the Littlewood-Offord problem, and in fact answer it in any number of dimensions (note that the case $k = 2$ deals with the special case of complex numbers).

Theorem 11. *Let $k, n \geq 1$ and suppose that $x_1, \ldots, x_n \in \mathbb{R}^k$ satisfy $||x_i||_2 \geq 1$ for all i. Let $K \subseteq \mathbb{R}^k$ have diameter $\mathrm{diam}(K) < 1$. Then there are at most $\binom{n}{\lfloor n/2 \rfloor}$ subsets $I \subseteq [n]$ such that $\sum_{i \in I} x_i \in K$.*

Proof. Let us define, for $A \subseteq [n]$, $x_A = \sum_{i \in A} x_i$. We shall call a family $\mathcal{A} \subseteq \mathcal{P}(n)$ *sparse* if

$$||x_A - x_B|| \geq 1$$

for all distinct $A, B \in \mathcal{A}$.

We shall say that a partition $D_1 \cup \cdots \cup D_m$ of $\mathcal{P}(n)$ is *symmetric* if it has the same number of sets of each size as a symmetric chain decomposition of $\mathcal{P}(n)$.

It is enough to show that $\mathcal{P}(n)$ has a symmetric partition into sparse families, as a sparse partition has $\binom{n}{\lfloor n/2 \rfloor}$ sets, and if I is a sparse set then there is at most one $A \in I$ with $x_A \in K$.

We prove the existence of a symmetric partition into sparse families by induction on n. The case $n = 1$ is easy. So suppose that $n > 1$ and that $\mathcal{F}_1, \ldots, \mathcal{F}_m$ is a symmetric partition of $\mathcal{P}(n-1)$ into sparse families (with respect to the vectors $x_1, \ldots, x_{n-1}$).

Suppose $\mathcal{F}_i = \{A_1, \ldots, A_t\}$, where

$$\langle x_{A_1}, x_n \rangle \leq \cdots \leq \langle x_{A_t}, x_n \rangle,$$

and $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^k$. Define two new families in $\mathcal{P}(n)$:

$$\mathcal{F}_i' := \{A_1, A_2, \ldots, A_t, A_t \cup n\}$$

and

$$\mathcal{F}_i'' := \{A_1 \cup n, A_2 \cup n, \ldots, A_{t-1} \cup n\},$$

discarding empty families (as in Proposition 10). We claim that this gives a symmetric partition of $\mathcal{P}(n)$ into sparse families.

It is clear that the construction gives the same number of sets of each size as in a symmetric chain decomposition of $\mathcal{P}(n)$; and that $\mathcal{F}_i''$ is sparse, since the corresponding sums are translations by x_n of the sums corresponding to $\mathcal{F}_i$. To see that $\mathcal{F}_i'$ is also sparse, we note that $\mathcal{F}_i$ is a sparse subset. Also, for any $j \in [r]$, we have (writing $\hat{x}_n = x_n/||x_n||_2$ for the unit vector in direction x_n)

$$\begin{aligned}
||x_{A_t \cup \{n\}} - x_{A_j}||_2 &\geq \langle x_{A_t \cup \{n\}} - x_{A_j}, \hat{x}_n \rangle \\
&= \langle x_{A_t \cup \{n\}}, \hat{x}_n \rangle - \langle x_{A_j}, \hat{x}_n \rangle \\
&= \langle x_{A_t} + x_n, \hat{x}_n \rangle - \langle x_{A_j}, \hat{x}_n \rangle \\
&\geq 1 + \langle x_{A_t}, \hat{x}_n \rangle - \langle x_{A_j}, \hat{x}_n \rangle \\
&\geq 1,
\end{aligned}$$

since $\langle x_{A_r}, x_n \rangle \geq \langle x_{A_j}, x_n \rangle$. ∎