Shadows and Kruskal-Katona

For $\mathcal{A} \subseteq [n]^{(r)}$, the Local LYM Inequality tells us that

$$\frac{|\partial\mathcal{A}|}{\binom{n}{r-1}} \geq \frac{|\mathcal{A}|}{\binom{n}{r}},$$

with equality iff $\mathcal{A} = \emptyset$ or $\mathcal{A} = [n]^r$. What happens in between?

It will be helpful to define two orders on $[n]^{(r)}$: the lexicographic and colexicographic orders.

In *lexicographic order* or *lex*, we have

$$A < B \text{ if } A \neq B \text{ and } \min(A \bigtriangleup B) \in A.$$

Equivalently, for distinct $A, B \in [n]^{(r)}$, with elements $a_1 < \cdots < a_r$ and $b_1 < \cdots < B_r$, we have $A < B$ if $a_i < b_i$, where $i = \min\{j : a_j \neq b_j\}$. This is the familiar dictionary order.

In *colexicographic order* or *colex*,

$$A < B \text{ if } A \neq B \text{ and } \max(A \bigtriangleup B) \in B.$$

Equivalently, we have $A < B$ if

$$\sum_{i \in A} 2^i < \sum_{i \in B} 2^i.$$

This can be thought of as 'binary' order.

We write $A <_{\text{lex}} B$ and $A <_{\text{colex}} B$ to distinguish betwen the two orders.

Lex and colex are very different. For instance, if we order pairs of natural numbers by colex we get

$$12, 13, 23, 14, 24, 34, 15, 25, 35, \ldots$$

while in colex we have

$$12, 13, 14, \ldots, 23, 24, 25, \ldots, 34, 35, 36, \ldots\ , \ldots.$$

The aim of this section is to prove the following theorem.

Theorem 12. (Kruskal-Katona Theorem) *Let $\mathcal{F} \subseteq [n]^{(r)}$ and let $\mathcal{A}$ be the family consisting of the first $|\mathcal{F}|$ elements of $[n]^{(r)}$ in colex order. Then* $|\partial\mathcal{F}| \geq |\partial\mathcal{A}|$.

In other words:

shadows are minimized by taking initial segments of colex.

Our strategy to prove the Kruskal-Katona Theorem is as follows: we replace $\mathcal{F}$ with a family $\mathcal{F}' \subseteq [n]^{(r)}$ such that

- $|\mathcal{F}'| = |\mathcal{F}|$
- $|\partial\mathcal{F}'| \geq |\partial\mathcal{F}|$
- $\mathcal{F}'$ is 'closer' to an initial segment of $[n]^{(r)}$.

We repeat, making the family 'nicer' at each step, and (hopefully) end up with an initial segment of colex.

In order to carry out this strategy, we will employ compression operators.[2] For distinct $i, j \in [n]$, the *compression operator* C_{ij} is the function from $\mathcal{P}(n)$ to $\mathcal{P}(n)$ defined by

$$C_{ij}(A) = \begin{cases} (A \setminus j) \cup i & \text{if } i \notin A, j \in A \\ A & \text{otherwise.} \end{cases}$$

For a set system $\mathcal{F}$, we define

$$C_{ij}(\mathcal{F}) := \{C_{ij}(A) : A \in \mathcal{F}\} \cup \{A \in \mathcal{F} : C_{ij}(A) \in \mathcal{F}\}.$$

If $i < j$, we sometimes refer to C_{ij} as a *left compression.*

Note that, for any $A \subseteq [n]$ and any $\mathcal{F} \subseteq \mathcal{P}(n)$, we have:

- $|C_{ij}(A)| = |A|$
- $|C_{ij}(\mathcal{F})| = |\mathcal{F}|$
- $C_{ij}(C_{ij}(\mathcal{F})) = C_{ij}(\mathcal{F})$
- if $j \in A$ and $A \in C_{ij}(\mathcal{F})$ then $A \in \mathcal{F}$ and $C_{ij}(A) \in \mathcal{F}$.

You should check all of these as an exercise!

If compressions are to be useful, we need to know that they interact well with shadows. This is indeed the case.

[2] Actually, these compression operators won't quite be enough to get what we want. But we will shortly define a slightly more general compression operator, and those *will* be enough.

Lemma 13. *For $1 \le i < j \le n$ and $\mathcal{F} \subseteq [n]^{(r)}$, we have $|\partial C_{ij}(\mathcal{F})| \le |\partial \mathcal{F}|$.*

Proof. Let $\mathcal{G} = C_{ij}(\mathcal{F})$: so we must show that $|\partial \mathcal{G}| \le |\partial \mathcal{F}|$. It will be enough to show the following.

Claim. *Let $G' \in \partial \mathcal{G} \setminus \partial \mathcal{F}$. Then*

1. $i \in G'$, $j \notin G'$
2. $(G' \setminus i) \cup j \in \partial \mathcal{F} \setminus \partial \mathcal{G}$.

If the claim holds, then it implies that C_{ji} gives an injection

$$C_{ji} : \partial \mathcal{G} \setminus \partial \mathcal{F} \to \partial \mathcal{F} \setminus \partial \mathcal{G}.$$

Indeed, (1) shows that C_{ji} is injective on $\partial G \setminus \partial \mathcal{F}$, and (2) shows that the image is contained in $\partial \mathcal{F} \setminus \partial \mathcal{G}$.

Thus we need only prove the claim. So consider $G' \in \partial \mathcal{G} \setminus \partial \mathcal{F}$. There are $G \in \mathcal{G}$ and $x \in [n]$ such that $G = G' \cup x$. Since $G' \in \partial \mathcal{G} \setminus \partial \mathcal{F}$, we must have $G \in \mathcal{G} \setminus \mathcal{F}$ and so $i \in G$ and $j \notin G$; we must also have $F := (G \setminus i) \cup j \in \mathcal{F} \setminus \mathcal{G}$.

If $x = i$ then $G' \subseteq F$, so we must have $x \ne i$. So $i \in G'$ and $j \notin G'$, which proves (1).

Let $F' = C_{ji}(G') = (G' \setminus i) \cup j$. Then $F' \subseteq F$, so $F' \in \partial \mathcal{F}$. All that remains is to show that

$$F' \notin \partial \mathcal{G}.$$

Suppose otherwise. Then there is z such that

$$F' \cup z = (G' \setminus i) \cup j \cup z \in \mathcal{G}.$$

Two cases:

- $z \ne i$: since $C_{ij}(\mathcal{G}) = \mathcal{G}$, we have

  $$C_{ij}(F' \cup z) = G' \cup z \in \mathcal{G}.$$

  But then $F' \cup z$ and $C_{ij}(F' \cup z)$ are both in $\mathcal{G}$, and so both are in $\mathcal{F}$. This gives a contradiction, as then $G' \subseteq C_{ij}(F' \cup z)$, so $G' \in \partial \mathcal{F}$.

- $z = i$: then $F' \cup z = G' \cup j \in \mathcal{G}$. Since $i, j \in G' \cup j$ we also have $G' \cup j \in \mathcal{F}$, which again gives a contradiction as then $G' \in \partial \mathcal{F}$.

We say that a family $\mathcal{F} \subseteq \mathcal{P}(n)$ is *left-compressed* if $C_{ij}(\mathcal{F}) = \mathcal{F}$ for all $1 \le i < j \le n$.

Corollary 14. *Let $\mathcal{F} \subseteq [n]^{(r)}$. Then there is a left-compressed family $\mathcal{A} \subseteq [n]^{(r)}$ such that $|\mathcal{A}| = |\mathcal{F}|$ and $|\partial\mathcal{A}| \le |\partial\mathcal{F}|$.*

Proof. For $\mathcal{A} \subseteq \mathcal{P}(n)$, we define the function

$$f(\mathcal{A}) := \sum_{A \in \mathcal{A}} \sum_{a \in A} 2^a.$$

Then for any $i < j$, applying C_{ij} either leaves $\mathcal{A}$ unchanged or strictly decreases the value of f.

Let $\mathcal{A} \subseteq \mathcal{P}(n)$ satisfy $|\mathcal{A}| = |\mathcal{F}|$, $|\partial\mathcal{A}| \le |\partial\mathcal{F}|$ and, subject to this, have $f(\mathcal{A})$ minimal. Then, by the lemma above, $\mathcal{A}$ must be left-compressed. ■

Any initial segment of $[n]^{(r)}$ in colex is left-compressed, so we might hope that Corollary 14 is enough to prove Kruskal-Katona. Unfortunately, not every left-compressed set system is an initial segment of colex: for instance $\{12, 13, 14\}$ is left-compessed but $23 <_{\text{colex}} 14$.

We will need a more general compression operator to prove Kruskal-Katona Theorem.

Let $U, V \subseteq [n]$ satisfy $|U| = |V|$ and $U \cap V = \emptyset$. The *UV-compression operator* C_{UV} is the function from $\mathcal{P}(n)$ to $\mathcal{P}(n)$ defined by

$$C_{UV}(A) = \begin{cases} (A \setminus V) \cup U & \text{if } U \cap A = \emptyset, V \subseteq A \\ A & \text{otherwise.} \end{cases}$$

For a set system $\mathcal{F}$, we define

$$C_{UV}(\mathcal{F}) := \{C_{UV}(A) : A \in \mathcal{F}\} \cup \{A \in \mathcal{F} : C_{UV}(A) \in \mathcal{F}\}.$$

A family $\mathcal{A}$ is *(U, V)-compressed* if $C_{UV}(\mathcal{A}) = \mathcal{A}$.

It is clear that $|C_{UV}(A)| = |A|$ and $|C_{UV}(\mathcal{A})| = |\mathcal{A}|$. We will use the following technical lemma, which extends Lemma 13.

Lemma 15. *Let $U, V \subseteq [n]$ be disjoint sets with $|U| = |V|$. Suppose that $\mathcal{F} \subseteq [n]^{(r)}$ satisfies*

$$\forall u \in U \ \exists v \in V \text{ such that } \mathcal{F} \text{ is } (U \setminus u, V \setminus v)\text{-compressed.} \tag{2.5}$$

Then $|\partial C_{UV}(\mathcal{F})| \le |\partial\mathcal{F}|$.

Proof. We generalize the proof of Lemma 13. Let $\mathcal{G} = C_{UV}(\mathcal{F})$ and consider $G' \in \partial\mathcal{G} \setminus \partial\mathcal{F}$. We will show:

Claim. *Let $G' \in \partial\mathcal{G} \setminus \partial\mathcal{F}$. Then*

1. $U \subseteq G'$, $V \cap G' = \emptyset$
2. $(G' \setminus V) \cup U \in \partial\mathcal{F} \setminus \partial\mathcal{G}$.

As before, this implies that C_{VU} gives an injection from $\partial\mathcal{G} \setminus \partial\mathcal{F}$ to $\partial\mathcal{F} \setminus \partial\mathcal{G}$, and the lemma follows.

Thus we need only prove the claim. So suppose we are given G' as in the claim. There are $G \in \mathcal{G}$ and $x \in [n]$ such that $G = G' \cup x$. Since $G' \in \partial\mathcal{G} \setminus \partial\mathcal{F}$, we must have $G \in \mathcal{G} \setminus \mathcal{F}$ and so $U \subseteq G$ and $V \cap G = \emptyset$; we must also have $F := (G \setminus U) \cup V \in \mathcal{F} \setminus \mathcal{G}$.

If $x \in U$ then by (2.5) there is $y \in V$ such that $\mathcal{F}$ is $(U \setminus x, V \setminus y)$-compressed, so

$$C_{U\setminus x, V\setminus y}(F) = (G \setminus x) \cup y \in \mathcal{F}.$$

But then $G' = G \setminus x \subseteq F$, which gives a contradiction. So we must have $x \notin U$. So $U \subseteq G'$ and $V \cap G' = \emptyset$, which proves (1).

Let $F' = C_{VU}(G') = (G' \setminus U) \cup V$. Then $F' \subseteq F$, so $F' \in \partial\mathcal{F}$. All that remains is to show that

$$F' \notin \partial\mathcal{G}.$$

Suppose otherwise. Then there is z such that

$$F' \cup z = (G' \setminus U) \cup V \cup z \in \mathcal{G}.$$

Two cases:

- $z \notin U$: since $C_{UV}(\mathcal{G}) = \mathcal{G}$, we have

  $$C_{UV}(F' \cup z) = G' \cup z \in \mathcal{G}.$$

  But then $F' \cup z$ and $C_{UV}(F' \cup z)$ are both in $\mathcal{G}$, and so both are in $\mathcal{F}$. This gives a contradiction, as then $G' \subseteq C_{UV}(F' \cup z)$, so $G' \in \partial\mathcal{F}$.

- $z \in U$: then $F' \cup z \in \mathcal{G}$ and $F' \cup z$ meets both U and V, so we must have $F' \cup z \in \mathcal{F}$. But there is $y \in V$ such that $\mathcal{F}$ is $(U \setminus u, V \setminus v)$-compressed, and so $C_{U\setminus u, V\setminus v}(F' \cup z) = G' \cup y \in \mathcal{F}$, which again gives a contradiction as then $G' \in \partial\mathcal{F}$.

We can now prove the Kruskal-Katona Theorem:

Proof of Kruskal-Katona. Let $\mathcal{A} \subseteq [n]^{(r)}$ satisfy

- $|\mathcal{A}| = |\mathcal{F}|$
- $|\partial\mathcal{A}| \leq |\partial\mathcal{F}|$
- subject to this, $\sum_{A\in\mathcal{A}}\sum_{i\in A} 2^i$ is minimal.

Let

$$\Lambda = \{(U, V) : |U| = |V| > 0, U \cap V \neq \emptyset, \max U < \max V\}.$$

If $\mathcal{A}$ is (U, V)-compressed for all $(U, V) \in \Lambda$ then $\mathcal{A}$ is an initial segment of colex: if $A \in \mathcal{A}$ and $B <_{\text{colex}} A$ then $\max(B \setminus A) < \max(A \setminus B)$ and so $\mathcal{A}$ is $(B \setminus A, A \setminus B)$-compressed, which implies $C_{B\setminus A, A\setminus B}(A) = B \in \mathcal{A}$.

Otherwise, pick $(U, V) \in \Lambda$ such that $\mathcal{A}$ is not (U, V)-compressed and $|U|$ is minimal. Then $\mathcal{A}$ is $(U \setminus u, V \setminus \min V)$-compressed for all $u \in U$, and so by Lemma 15 we have $|\partial C_{UV}(\mathcal{A})| \leq |\partial\mathcal{A}|$. But $C_{UV}(\mathcal{A})$ has strictly smaller weight than $\mathcal{A}$, which contradicts the minimality of the weight of $\mathcal{A}$. ■