Intersections and traces

Erdős-Ko-Rado and the Two Families Theorem

A family $\mathcal{A} \subseteq \mathcal{P}(n)$ is *intersecting* if $|A \cap B| \neq \emptyset$, for all $A, B \in \mathcal{A}$.

What is the maximum size of an intersecting family in $\mathcal{P}(n)$? The set $\{A \subseteq [n] : 1 \in A\}$ is intersecting and has size 2^{n-1}. It is easy to show that this is best possible.

Proposition 16. *Let $\mathcal{A} \subseteq \mathcal{P}(n)$ be intersecting. Then $|\mathcal{A}| \leq 2^{n-1}$.*

Proof. $\mathcal{A}$ contains at most one set from each pair $(A, [n] \setminus A)$. ∎

A much more interesting question is: what is the largest intersecting family of r-sets in $\mathcal{P}(n)$? There are three regimes to consider:

- $r > n/2$: This case is trivial, as we can take the entire layer $[n]^{(r)}$.

- $r = n/2$: (Obviously, this only happens when n is even.) This case is easy: we can take at most one from each pair $(A, [n] \setminus A)$, and any system obtained in this way is intersecting. Thus the maximum is

$$\frac{1}{2}\binom{n}{n/2} = \binom{n-1}{n/2-1} = \binom{n-1}{r-1}.$$

- $r < n/2$: This is more interesting! One example is to take all r-sets containing a fixed element, say 1. This gives a system of size $\binom{n-1}{r-1}$.

> Very good. But we could also take all sets that contain at least two elements from $\{1,2,3\}$. This also gives an intersecting system, and a litle calculation shows that it has size $3\binom{n-3}{r-2}+\binom{n-3}{r-3}$. A little more calculation shows that the first system is bigger, but of course we want to handle all possible systems.

This last case is where the Erdős-Ko-Rado Theorem comes in.

Theorem 17. *(Erdős-Ko-Rado Theorem) For $r \leq n/2$, if $\mathcal{A} \subseteq [n]^{(r)}$ is intersecting then $|\mathcal{A}| \leq \binom{n-1}{r-1}$.*

Remark: In fact, for $r < n/2$, we get equality only for the systems that consist of all r-sets containing a fixed point. (We won't prove this here.) For $r = n/2$ it is easy to construct many nonisomorphic systems.

We shall give two proofs: the first uses Katona's ingenious circle method; the second uses the Kruskal-Katona Theorem.

First proof of the Erdős-Ko-Rado Theorem. Consider any bijection $f : [n] \to \mathbb{Z}_n$. We say that A maps to an interval under f if $f(A) := \{f(a) : a \in A\} = \{i, i+1, \ldots, i+k-1\}$, for some $0 \leq i \leq n-1$ (where addition is modulo n). We will double count the number N of pairs (f, A) such that $f : [n] \to \mathbb{Z}_n$ is a bijection and $f(A)$ is an interval.

For any fixed f, we claim that at most k sets in $\mathcal{A}$ map to intervals under f. Indeed, suppose $A \in \mathcal{A}$ and $f(A) = \{i, i+1, \ldots, i+k-1\}$. Since $\mathcal{A}$ is intersecting, any other interval that we get under f must be of form

$$\{j, j-1, \ldots, j-(k-1)\}$$

or

$$\{j+1, j+2, \ldots, j+k\}),$$

for some $j \in \{i, i+1, \ldots, i+k-2\}$. But for each j we can get at most one of these two intervals (as they are disjoint). So we get at most $k-1$ such intervals, and hence at most k in total. Summing over all $n!$ bijections from $[n]$ to $\mathbb{Z}_n$, we see that

$$N \leq kn!.$$

On the other hand, each $A \in \mathcal{A}$ is an interval under $n(n-k)!k!$ bijections, so

$$N = |\mathcal{A}|n(n-k)!k!.$$

Combining these bounds on N gives

$$|\mathcal{A}| \leq \frac{kn!}{n(n-k)!k!} = \binom{n-1}{k-1}.$$

■

Now for a proof involving shadows. Recall that $\partial\mathcal{A}$ is the shadow of the set system $\mathcal{A}$. We write $\partial^{(2)}\mathcal{A} = \partial(\partial A)$, $\partial^{(3)}\mathcal{A} = \partial(\partial^{(2)}A)$, and so on. Note that $\partial^{(k)}\mathcal{A}$ is the collection of sets B that can be obtained from some $A \in \mathcal{A}$ by deleting k elements.

Second proof of the Erdős-Ko-Rado Theorem. Let $\mathcal{A} \subseteq [n]^{(r)}$ be an intersecting family, and set

$$\mathcal{B} = \{A^c : A \in \mathcal{A}\} \subseteq [n]^{(n-r)}.$$

Since $\mathcal{A}$ is intersecting, no set $A \in \mathcal{A}$ is contained in any set $B \in \mathcal{B}$. So

$$\partial^{n-2r}\mathcal{B} \subseteq [n]^{(r)}$$

is disjoint from $\mathcal{A}$.

Now if $|\mathcal{A}| \geq \binom{n-1}{r-1}$ then

$$|\mathcal{B}| = |\mathcal{A}| \geq \binom{n-1}{r-1} = \binom{n-1}{n-r}.$$

We now apply Kruskal-Katona repeatedly:

$$|\partial\mathcal{B}| \geq |\partial[n-1]^{(n-r)}| = \binom{n-1}{n-r-1}$$

and so

$$|\partial^{(2)}\mathcal{B}| \geq |\partial[n-1]^{(n-r-1)}| = \binom{n-1}{n-r-2},$$

and so on, showing at each step that $|\partial^{(i)}\mathcal{B}| \geq \binom{n-1}{n-r-i}$, until we get

$$|\partial^{(n-2r)}\mathcal{B}| \geq |\partial[n-1]^{(n-r-1)}| = \binom{n-1}{n-r-2}.$$

So if $|\mathcal{A}| > \binom{n-1}{r-1}$, we get

$$\binom{n}{r} \geq |\mathcal{A} \cup \mathcal{B}| > \binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r},$$

which gives a contradiction. ■

We next prove the Two Families Theorem, which is due to Bollobás.

Theorem 18. (Two Families Theorem) *Let $A_1, \dots, A_k$ and $B_1, \dots, B_k$ be finite sets such that, for all i,*

$$A_i \cap B_i = \emptyset$$

and, for all $i \neq j$,

$$A_i \cap B_j \neq \emptyset.$$

Then

$$\sum_{i=1}^{k} \binom{|A_i| + |B_i|}{|A_i|}^{-1} \leq 1.$$

If we specify the size of the sets, we get the following useful corollary.

Corollary 19. *Let $A_1, \dots, A_k$ be a-sets and $B_1, \dots, B_k$ be b-sets such that, for all i,*

$$A_i \cap B_i = \emptyset$$

and, for all $i \neq j$,

$$A_i \cap B_j \neq \emptyset.$$

Then

$$k \leq \binom{a+b}{a}.$$

Corollary 19 is an immediate consequence of the Two Families Theorem.

Proof of the Two Families Theorem. We may assume that all sets are subsets of $[n]$. For a permutation π of $[n]$, we write $A <_\pi B$ if

$$\max \pi(A) < \min \pi(B),$$

where we write $\pi(S) := \{\pi(x) : x \in S\}$.

Let $\pi \in S_n$ be chosen uniformly at random from the set of permutations of $[n]$. Then, for each i, as $A_i \cap B_i = \emptyset$, we have

$$\mathbb{P}(A_i <_\pi B_i) = \binom{|A_i| + |B_i|}{|A_i|}^{-1},$$

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$$\mathbb{P}(A_i <_\pi B_i) = \binom{|A_i| + |B_i|}{|A_i|}^{-1},$$

On the other hand, if $A_i <_\pi B_i$ then $A_j \not< B_j$ for $j \neq i$ (as $A_i \cap B_j$ and $A_j \cap B_i$ are both nonempty). So the events $(A_i < B_i)_{i\in[k]}$ are disjoint, and so

$$1 \geq \sum_{i=1}^{k} \mathbb{P}(A_i <_\pi B_i) = \sum_{i=1}^{k} \binom{|A_i| + |B_i|}{|A_i|}^{-1},$$

which gives the required inequality. ■

Let us see an application of the Two Families Theorem.

An r*-uniform hypergraph* $H = (V, E)$ consists of a set V (of *vertices*) and a set $E \subseteq V^{(r)}$ (of *edges*). The *complete* r*-uniform hypergraph on* k *vertices* is $K_k^{(r)} := ([k], [k]^{(r)})$. Isomorphism is defined just as you expect. If H' is isomorphic to H we will often say that H' is a *copy* of H.

Let H be an r-uniform hypergraph. We say that an r-uniform hypergraph G is H-saturated if G does not contain a copy of H, but if we add any edge to G then the resulting hypergraph contains a copy of $\mathcal{H}$.

For instance, in the case of graphs, if $H = K_3$ then Turán's Theorem tells us that the mamximal number of edges in an H-saturated graph on n vertices is $\lfloor n^2/4 \rfloor$; it is an exercise to show that the minimum number of edges is $n - 1$.

In general, it is very hard to determine the maximum number of edges in a $K_k^{(r)}$-saturated hypegraph. Suprisingly, Bollobás determined the *minimum* number precisely.

Theorem 20. *Let G be an r-uniform hypergraph with vertex set $[n]$, and suppose that adding any edge to G creates a copy of $K_r^{(r+s)}$. Then G has at least*

$$\binom{n}{r} - \binom{n-s}{r}$$

edges.

Proof. Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be the *non-edges* of G. For each i, there is an $(r+s)$-element set $K_i \supset A_i$ such that adding A_i to $\mathcal{G}$ creates a copy of $K_{r+s}^{(r)}$ with vertex set K_i. Let $B_i = [n] \setminus K_i$. Then

- $|A_i| = r$ and $|B_i| = n - r - s$ for each i;
- $A_i \cap B_i = \emptyset$ for each i;

- for distinct i, j, we have $A_i \cap B_i \neq \emptyset$ (or else we would have $A_i \subseteq [n] \setminus B_j = K_j$, and so G would be mising two edges A_i, A_j from the complete r-graph with vertex set K_i).

So we can apply the Two Families Theorem to get

$$m \leq \binom{r + (n - r - s)}{r} = \binom{n - s}{r}.$$

■

The above bound is sharp: we can take the r-uniform hypergraph with vertex class $[n]$ and edges $\{F \in [n](r) : F \cap [s] \neq \emptyset\}$.