

INTERSECTING FAMILIES CONTINUED

**Theorem 27** (Frankl-Wilson Theorem). *Let  $\mathcal{A} \subseteq \mathcal{P}(n)$  be  $S$ -intersecting. Then*

$$|\mathcal{A}| \leq \sum_{i=0}^{|S|} \binom{n}{i}.$$

We won't prove the full version of F-W here: instead we prove it under the additional assumption that  $|A| \notin S$  for all  $A \in \mathcal{A}$ .

*Proof.* This is an easy deduction from the Modular Frankl-Wilson Theorem: just apply the modular version of the theorem with  $p > n$ . ■

The next result is a uniform version of the Modular Frankl-Wilson Theorem.

**Theorem 28** (Ray-Chaudhuri-Wilson Theorem). *Let  $\mathcal{A} \subseteq [n]^{(k)}$  be an  $S$ -intersecting system, where  $S \subseteq \{0, \dots, k-1\}$ . Then*

$$|\mathcal{A}| \leq \binom{n}{|S|}.$$

*Proof.* Let  $x = (x_1, \dots, x_n)$ . For each  $A \in \mathcal{A}$ , we define

$$f_A(x) = \prod_{s \in S} \left( \sum_{i \in A} x_i - s \right).$$

So for  $A, B \in \mathcal{A}$  we have

$$f_A(\chi_B) = \prod_{s \in S} (|A \cap B| - s) = \begin{cases} 0 & A \neq B \\ \neq 0 & A = B. \end{cases}$$

Now for  $B \subseteq [n]$ , let

$$p_B(x) = \prod_{i \in B} x_i.$$

(In particular  $p_\emptyset = 1$ .) We have

$$p_A(\chi_B) = \begin{cases} 1 & A \subseteq B \\ 0 & A \not\subseteq B. \end{cases}$$

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Finally, define

$$q(x) = \sum_{i=0}^n x_i - k,$$

so  $q(\chi_B) = 0$  for  $B \in \mathcal{A}$ , and  $q(\chi_B) \neq 0$  if  $|B| < k$ .

Define the collection

$$\mathcal{G} : \quad \{f_A : A \in \mathcal{A}\} \cup \{q \cdot p_B : |B| \leq |S| - 1\},$$

and let  $\tilde{\mathcal{G}}$  be the multilinearized collection

$$\tilde{\mathcal{G}} : \quad \{\tilde{f}_A : A \in \mathcal{A}\} \cup \{\widetilde{qp}_B : |B| \leq |S| - 1\}.$$

We claim that the collection  $\tilde{\mathcal{G}}$  is linearly independent. If this is true then note that all the polynomials in  $\tilde{\mathcal{G}}$  are multilinear and have degree at most  $s$ . Thus they lie in a space of dimension  $\sum_{i=0}^{|S|} \binom{n}{i}$ . However, the set  $\{\widetilde{qp}_B : |B| \leq |S| - 1\}$  has size  $\sum_{i=0}^{|S|-1} \binom{n}{i}$ . So the set  $\mathcal{A}$  has size at most  $\binom{n}{|S|}$ .

All that remains is to prove our claim. Suppose that

$$\sum_{A \in \mathcal{A}} \lambda_A \tilde{f}_A + \sum_{|B| \leq |S| - 1} \mu_B \widetilde{qp}_B = 0.$$

For  $C \in \mathcal{A}$ , we have

$$\widetilde{qp}_B(\chi_C) = qp_B(\chi_C) = 0,$$

so for all  $C \in \mathcal{A}$  we have

$$0 = \left( \sum_{A \in \mathcal{A}} \lambda_A \tilde{f}_A + \sum_{|B| \leq |S| - 1} \mu_B \widetilde{qp}_B \right) (\chi_C) = \lambda_C.$$

It follows that we must have

$$\sum_{|B| \leq |S| - 1} \mu_B \widetilde{qp}_B = 0.$$

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Now if not all  $\mu_C$  are 0 then let  $C$  be of minimal size with  $\mu_C \neq 0$ . Then

$$\mu_B \widetilde{qp}_B(\chi_C) = \begin{cases} 0 & B \subseteq C, B \neq C \text{ (as } \mu_B = 0\text{)} \\ 0 & B \not\subseteq C \text{ (as } p_B(\chi_C) = 0\text{)} \\ \neq 0 & B = C. \end{cases}$$

So

$$0 = \sum_{|B| \leq |S|-1} \mu_B \widetilde{qp}_B(\chi_C) = \mu_C \widetilde{qp}_C(\chi_C) \neq 0,$$

which gives a contradiction. Thus the claim holds.