Borsuk's Conjecture

In 1933, Borsuk conjectured that if $K \subseteq \mathbb{R}^d$ has diameter 1 then it can be partitioned into $d+1$ sets of diameter less than 1. (It's easy to see that there are sets for which $d+1$ is necessary: consider the regular simplex.)

Borsuk's conjecture remained open until 1993, when Kahn and Kalai showed that it is false.

Theorem 29. *Let $k(d)$ be the smallest integer k such that every subset of $\mathbb{R}^n$ of diameter 1 can be partitioned into $d+1$ sets of smaller diameter. Then there exists $c > 1$ such that*

$$k(d) \geq c^{\sqrt{d}}$$

for infinitely many n.

In fact, Kahn and Kalai proved that (with a little more work) the bound holds for all d. Surprisingly, their result uses the Modular Frankl-Wilson Theorem.

We will need a preliminary lemma.

Lemma 30. *Let p be a prime and suppose $\mathcal{A} \subseteq [4p]^{(2p)}$. Suppose there is no pair of distinct $A, B \in \mathcal{A}$ with $|A \cap B| = p$. Then $|\mathcal{A}| \leq 4\binom{4p}{p-1}$.*

Proof. For $x \in [4p]$, let $\mathcal{A}_x := \{A \in \mathcal{A} : x \in A\}$. We can choose x such that $|\mathcal{A}_x| \geq \frac{1}{2}|\mathcal{A}|$. Then, for distinct $A, B \in \mathcal{A}_x$, we have $|A \cap B| \neq 0, p, 2p$. So, for distinct $A, B \in \mathcal{A}_x$, we have

$$|A \cap B| \not\equiv 0 \bmod p.$$

On the other hand, for all $A \in \mathcal{A}_x$,

$$|A| \equiv 0 \bmod p.$$

We can therefore apply the Modular Frankl-Wilson Theorem (with $S = \{1, \ldots, p-1\}$) to get

$$|\mathcal{A}_x| \leq \sum_{i=0}^{p-1} \binom{4p}{i} \leq 2\binom{4p}{p-1},$$

since $2\binom{4p}{i} \geq \binom{4p}{i-1}$, for all $i \leq p-1$. The result follows. ■

Proof of the Kahn-Kalai Theorem. Let $N = \binom{4p}{2}$, where p is a prime. We shall construct a set $K \subseteq \mathbb{R}^n$. In fact, let

$$W = [4p]^{(2)} = E(K_{4p}),$$

so we identify $[4p]^{(2)}$ with the edges of the complete graph with vertex set $[4p]$. We will work in $\mathbb{R}^W$. Note that

- Coordinates in $\mathbb{R}^W$ are indexed by edges of K_{4p}; and
- 0-1 vectors correspond to subgraphs of K_{4p}.

For each $A \in [4p]^{(2p)}$, let

$$E_A = \{\{i,j\} \in W : |A \cap \{i,j\}| = 1\};$$

in other words E_A is the edge set of the complete bipartite graph with vertex classes A and A^c. We set

$$\mathcal{F} = \{E_A : A \in [4p]^{(2p)}\}.$$

Since $|A| = |A^c| = 2p$ and $E_A = E_{A^c}$, we have

$$|\mathcal{F}| = \frac{1}{2}\binom{4p}{2p}.$$

We identify each element E_A of $\mathcal{F}$ with the vector v_A given by

$$(v_A)_e = \begin{cases} 1 & e \in \mathcal{F} \\ 0 & e \notin \mathcal{F}. \end{cases}$$

Let K be the set of points in $\mathbb{R}^W$ corresponding to $\mathcal{F}$.

Now for $A, B \in [4p]^{(2p)}$,

$$\begin{aligned} ||v_A - v_B||_2 &= \left(\sum_e (v_A(e) - v_B(e))^2\right)^{1/2} \\ &= (|E_A| + |E_B| - 2|E_A \cap E_B|)^{1/2}. \end{aligned}$$

So $||v_A - v_B||_2$ increases as $|E_A \cap E_B|$ decreases. It is easy to check that $|E_A \cap E_B|$ is minimal if and only if $|A \cap B| = p$.

Now suppose $L \subseteq K$ satisfies $\mathrm{diam}(L) < \mathrm{diam}(K)$. In the set $\mathcal{A} \subseteq \mathcal{F}$ corresponding to L there can be no pair A, B with $|A \cap B| = p$. So by the lemma, $|\mathcal{A}| \leq \binom{4p}{p-1}$, and so

$$|\mathcal{F}|/|\mathcal{A}| \geq \frac{1}{2}\binom{4p}{2p}\Big/\binom{4p}{p-1}.$$

For large p, an application of Stirling's formula shows that this is at least 1.6^p, which is at least $1.5^{\sqrt{n}}$. Thus if we want to partition K into sets of smaller diameter we need at least $1.5^{\sqrt{n}}$ sets. ■