Chapter 2 - Similitude (Keyword: EQUAL RATIOS)

Similitude: Similarity of behavior for different systems with equal similarity parameters.

Prototype	$\leftrightarrow$	Model
(real world)		(physical/ analytical/ numerical … experiments)

Similitude	Similarity Parameters (SP's)
Geometric Similitude	Length ratios, angles
Kinematic Similitude	Displacement ratios, velocity ratios
Dynamic Similitude	Force ratios, stress ratios, pressure ratios
⋮	
Internal Constitution Similitude	ρ, ν
Boundary Condition Similitude	
⋮	

For similitude we require that the similarity parameters SP's (eg. **angles**, length **ratios**, velocity **ratios**, etc) are equal for the model and the real world.

Example 1 Two similar triangles have equal **angles** or equal **length ratios**. In this case the two triangles have *geometric similitude.*

Example 2 For the flow around a model ship to be similar to the flow around the prototype ship, both model and prototype need to have equal **angles** and equal **length** and **force ratios**. In this case the model and the prototype have *geometric and dynamic similitude.*

2.1 Dimensional Analysis (DA) to Obtain SP's

2.1.1 Buckingham's π theory

Reduce number of variables → derive dimensionally homogeneous relationships.

1. Specify (all) the (say N) relevant variables (dependent or independent): $x_1, x_2, \ldots x_N$
   e.g. time, force, fluid density, distance…
   We want to relate the x_i's to each other $\mathcal{I}(\ x_1, x_2, \ldots x_N) = 0$

2. Identify (all) the (say P) relevant basic physical units ("dimensions")
   e.g. M,L,T (P = 3) [temperature, charge, …].

3. Let $\pi = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N}$ be a dimensionless quantity formed from the x_i's. Suppose

$$x_i = C_i M^{m_i} L^{l_i} T^{t_i}, i = 1, 2, \ldots, N$$

   where the C_i are dimensionless constants. For example, if $x_1 = KE = \frac{1}{2}MV^2 = \frac{1}{2}M^1L^2T^{-2}$ (kinetic energy), we have that $C_1 = \frac{1}{2}, m_1 = 1, l_1 = 2, t_1 = -2$. Then

$$\pi = (C_1^{\alpha_1} C_2^{\alpha_2} \ldots C_N^{\alpha_N}) M^{\alpha_1 m_1 + \alpha_2 m_2 + \ldots + \alpha_N m_N} L^{\alpha_1 l_1 + \alpha_2 l_2 + \ldots + \alpha_N l_N} T^{\alpha_1 t_1 + \alpha_2 t_2 + \ldots + \alpha_N t_N}$$

   For π to be dimensionless, we require

$$P \left\{ \begin{array}{c} \overbrace{\begin{array}{c} \alpha_i m_i = 0 \\ \alpha_i l_i = 0 \\ \alpha_i t_i = 0 \end{array}}^{N} \\[-4pt] \underbrace{\hphantom{\alpha_i m_i = 0}}_{\Sigma \text{ notation}} \end{array} \right\} \quad \text{a} P \times N \text{ system of Linear Equations} \tag{1}$$

   Since (1) is homogeneous, it always has a trivial solution,

$$\alpha_i \equiv 0, i = 1, 2, \ldots, N \text{ (i.e. } \pi \text{ is constant)}$$

   There are 2 possibilities:

   (a) (1) has no nontrivial solution (only solution is π = constant, i.e. independent of x_i's), which implies that the N variable $x_i, i = 1, 2, \ldots, N$ are Dimensionally Independent (DI), i.e. they are 'unrelated' and 'irrelevant' to the problem.

   (b) (1) has $J (J > 0)$ nontrivial solutions, $\pi_1, \pi_2, \ldots, \pi_J$. In general, $J < N$, in fact, $J = N - K$ where K is the rank or 'dimension' of the system of equations (1).

2.1.2 **Model Law**

Instead of relating the N x_i's by $\mathcal{I}(x_1, x_2, \ldots x_N) = 0$, relate the J π's by

$$F(\pi_1, \pi_2, \ldots \pi_J) = 0, \text{ where } J = N - K < N$$

For similitude, we require

$$(\pi_{\text{model}})_j = (\pi_{\text{prototype}})_j \text{ where } j = 1, 2, \ldots, J.$$

If 2 problems have all the same π_j's, they have similitude (in the π_j senses), so π's serve as similarity parameters.

Note:

- If π is dimensionless, so is $\pi \times const$, π^{const}, $1/\pi$, etc...
- If π_1, π_2 are dimensionless, so is $\pi_1 \times \pi_2$, $\frac{\pi_1}{\pi_2}$, $\pi_1^{const_1} \times \pi_2^{const_2}$, etc...

In general, we want the set (not unique) of independent π_j's, for e.g., π_1, π_2, π_3 or π_1, $\pi_1 \times \pi_2$, π_3, but not π_1, π_2, $\pi_1 \times \pi_2$.

Example: Force on a smooth circular cylinder in steady, incompressible flow
Application of Buckingham's π Theory.

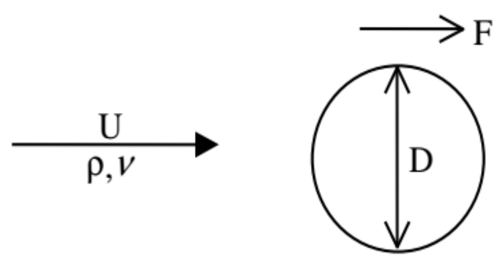

Figure 1: Force on a smooth circular cylinder in steady incompressible fluid (no gravity)

A Fluid Mechanician found that the relevant *dimensional* quantities required to evaluate the force F on the cylinder from the fluid are: the diameter of the cylinder D, the fluid velocity U, the fluid density ρ and the kinematic viscosity of the fluid ν. Evaluate the *non-dimensional* independent parameters that describe this problem.

$$x_i : F, U, D, \rho, \nu \;\rightarrow\; N = 5$$
$$x_i = c_i M^{m_i} L^{l_i} T^{t_i} \;\rightarrow\; P = 3$$

		F	U	D	ρ	ν
		$N = 5$				
P = 3	m_i	1	0	0	1	0
	l_i	1	1	1	-3	2
	t_i	-2	-1	0	0	-1

$$\pi = F^{\alpha_1} U^{\alpha_2} D^{\alpha_3} \rho^{\alpha_4} \nu^{\alpha_5}$$

For π to be non-dimensional, the set of equations

$$\begin{aligned} \alpha_i m_i &= 0 \\ \alpha_i l_i &= 0 \\ \alpha_i t_i &= 0 \end{aligned}$$

has to be satisfied. The system of equations above after we substitute the values for the m_i's, l_i's and t_i's assume the form:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & -3 & 2 \\ -2 & -1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The rank of this system is $K = 3$, so we have $j = 2$ nontrivial solutions. Two families of solutions for α_i for each fixed pair of $(\alpha_4,\ \alpha_5)$, exists a unique solution for $(\alpha_1,\ \alpha_2,\ \alpha_3)$. We consider the pairs $(\alpha_4 = 1,\ \alpha_5 = 0)$ and $(\alpha_4 = 0,\ \alpha_5 = 1)$, all other cases are linear combinations of these two.

1. Pair $\alpha_4 = 1$ and $\alpha_5 = 0$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$$

which has solution

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

$$\therefore \pi_1 = F^{\alpha_1} U^{\alpha_2} D^{\alpha_3} \rho^{\alpha_4} \nu^{\alpha_5} = \frac{\rho U^2 D^2}{F}$$

Conventionally, $\pi_1 \to 2\pi_1^{-1}$ and $\therefore \pi_1 = \frac{F}{\frac{1}{2}\rho U^2 D^2} \equiv C_d$, which is the Drag coefficient.

2. Pair $\alpha_4 = 0$ and $\alpha_5 = 1$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$$

which has solution

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

$$\therefore \pi_2 = F^{\alpha_1} U^{\alpha_2} D^{\alpha_3} \rho^{\alpha_4} \upsilon^{\alpha_5} = \frac{\nu}{UD}$$

Conventionally, $\pi_2 \to \pi_2^{-1}, \therefore \pi_2 = \frac{UD}{\nu} \equiv R_e$, which is the Reynolds number.

Therefore, we can write the following equivalent expressions for the *non-dimensional* independent parameters that describe this problem:

$$\begin{aligned} F(\pi_1, \pi_2) &= 0 & \text{or} && \pi_1 &= f(\pi_2) \\ F(\mathcal{C}_d, \mathcal{R}_e) &= 0 & \text{or} && \mathcal{C}_d &= f(\mathcal{R}_e) \\ F(\frac{F}{1/2\ \rho U^2 D^2}, \frac{UD}{\nu}) &= 0 & \text{or} && \frac{F}{1/2\ \rho U^2 D^2} &= f(\frac{UD}{\nu}) \end{aligned}$$

Appendix A

Dimensions of *some* fluid properties

Quantities		Dimensions (MLT)
Angle	θ	none ($M^0L^0T^0$)
Length	L	L
Area	A	L^2
Volume	$\forall$	L^3
Time	t	T
Velocity	V	LT^{-1}
Acceleration	$\dot{V}$	LT^{-2}
Angular velocity	ω	T^{-1}
Density	ρ	ML^{-3}
Momentum	$\mathcal{L}$	MLT^{-1}
Volume flow rate	$\mathcal{Q}$	L^3T^{-1}
Mass flow rate	$\mathcal{Q}$	MT^{-1}
Pressure	p	$ML^{-1}T^{-2}$
Stress	τ	$ML^{-1}T^{-2}$
Surface tension	Σ	MT^{-2}
Force	F	MLT^{-2}
Moment	M	ML^2T^{-2}
Energy	E	ML^2T^{-2}
Power	P	ML^2T^{-3}
Dynamic viscosity	μ	$ML^{-1}T^{-1}$
Kinematic viscosity	ν	L^2T^{-1}