BIOLOGICAL CONTROL SYSTEMS

SYSTEM CONCEPTS

SIGNAL FLOW

SIGNAL FLOW GRAPH

SFG is a diagram that represents a set of simultaneous linear algebraic equations which describe a system. Let us consider an equation, $Y = aX$. It may be represented graphically as,

where 'a' is called ***transmittance*** or transmission function.

Definitions in SFG

Node – A system variable, the value of which equals the sum of all incoming signals at the node.

Branch – A directed line segment joining two nodes.

Input/ Output node – node having only one outgoing/ incoming branch.

Path – A traversal of connected branches in the direction of branch arrows.

Forward path – A path from input to output node.

Loop – A closed path that originates and terminates on the same node.

Self-loop – A loop containing one branch.

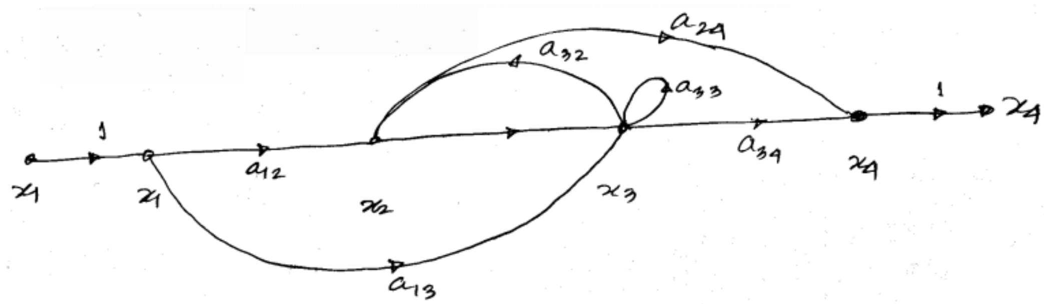
Non-touching loops – Loops which do not have a common node.

Gain – Transmittance of a branch.

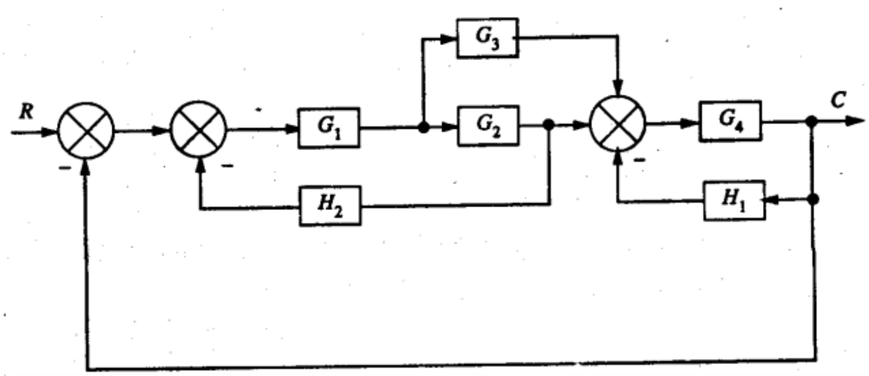
Construction of SFGs

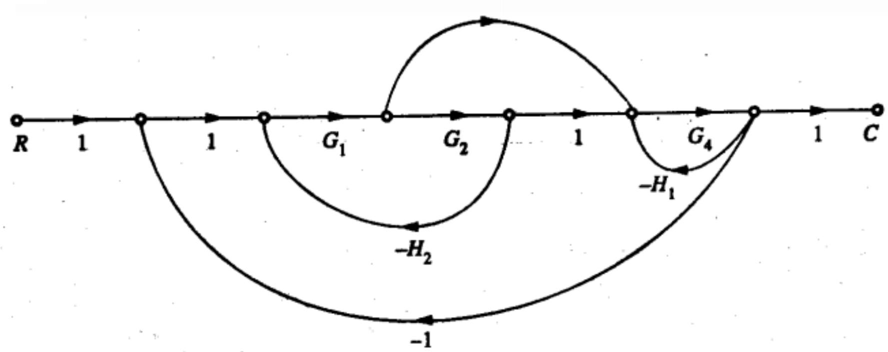
The SFG of a system can be constructed from the describing equations:

$$x_2 = a_{12}x_1 + a_{32}x_3$$
$$x_3 = a_{13}x_1 + a_{23}x_2 + a_{33}x_3$$
$$x_4 = a_{24}x_4 + a_{34}x_3$$



SFG from Block Diagram





Each variable in the block diagram becomes a node, and each block becomes a branch.

Mason’s Gain Formulae

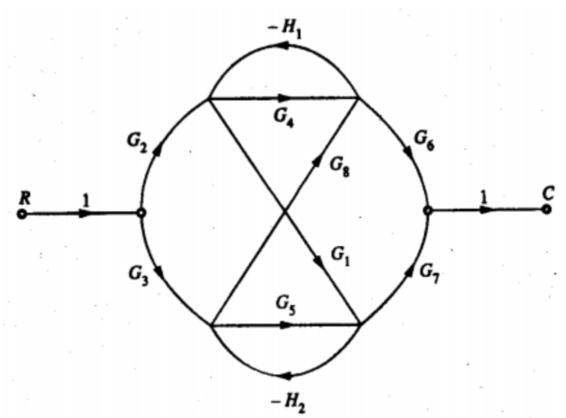
It is possible to write the overall transfer function of a system through inspection of SFG using Mason's gain formulae given by, $T = (\sum_i P_i \Delta_i)/\Delta$.

where T = overall gain of the system, P_i = path gain of *ith* forward path, Δ = determinant of SFG, Δ_i = value of Δ for that part of the graph not touching the *ith* forward path.

$\Delta = 1 - \sum_j P_{j1} + \sum_j P_{j2} - \sum_j P_{j3} + \cdots$ = 1 – [sum of loop gain of all individual loops] + [sum of all gain-products of two non-touching loops] – [sum of all gain-products of three non-touching loops] + ……;

$P_{jk} = j$th product of k non-touching loops.

Example



1. There are 6 forward paths with path gains

$$P_1 = G_2 G_4 G_6$$
$$P_2 = G_3 G_5 G_7$$
$$P_3 = G_2 G_1 \cdot G_7$$
$$P_4 = G_3 G_8 G_6$$
$$P_5 = -G_2 G_1 \cdot H_2 G_8 \cdot G_6$$
$$P_6 = -G_3 G_8 H_1 G_1 G_7$$

2. There are three individual loops with loop gains

$$P_{11} = -H_1 G_4$$
$$P_{21} = -H_2 G_5$$
$$P_{31} = G_1 H_2 G_8 H_1$$

3. There is only one combination of two non-touching loops

$$P_{12} = H_1 H_2 G_4 G_5$$

4. There are no combinations of more than two non-touching loops.

5. Hence,

$$\Delta = 1 - [-H_1 G_4 - H_2 G_5 + G_1 H_2 G_8 H_1] + [H_1 H_2 G_4 G_5]$$
$$= 1 - G_1 H_2 G_8 H_1 + H_2 G_5 - G_1 H_2 G_8 H_1 + H_1 H_2 G_4 G_5$$

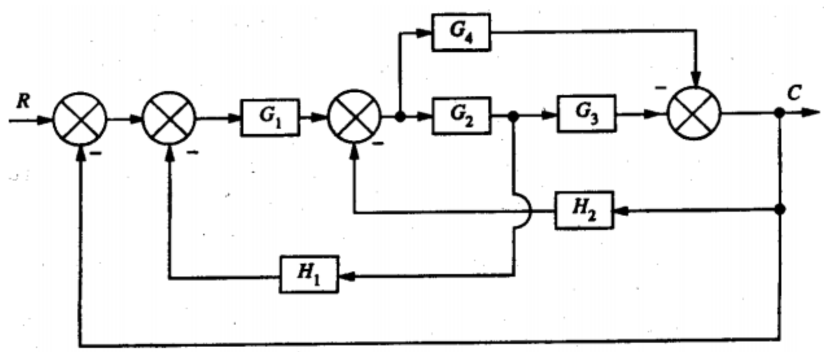
$\Delta_1 = 1 - (-H_2 G_5) = 1 + H_2 G_5$; $\Delta_2 = 1 - (-H_1 G_4) = 1 + H_1 G_4$;

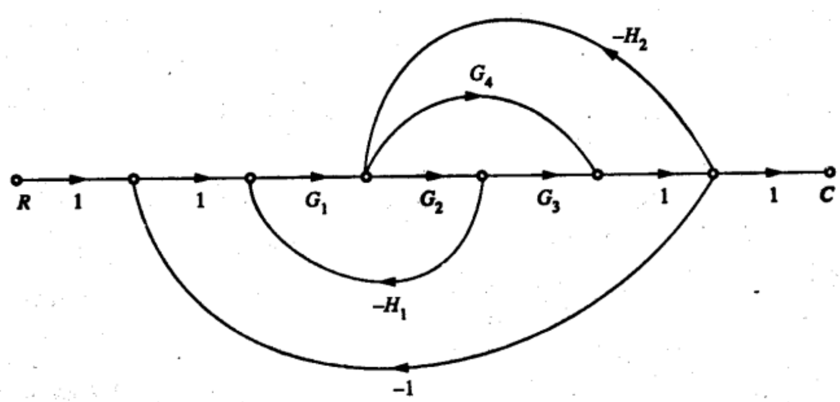
$\Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 1$

Thus, $T = \dfrac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4 + P_5\Delta_5 + P_6\Delta_6}{\Delta}$, where P_1, Δ_1, Δ etc. are derived before.

Example

Draw the SFG and determine C/ R for the block diagram shown in Figure below.





$$\frac{C}{R} = \frac{G_1 G_2 G_3 + G_1 G_4}{1 + G_1 G_2 G_3 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_4 + G_4 H_2}$$ {Answer}

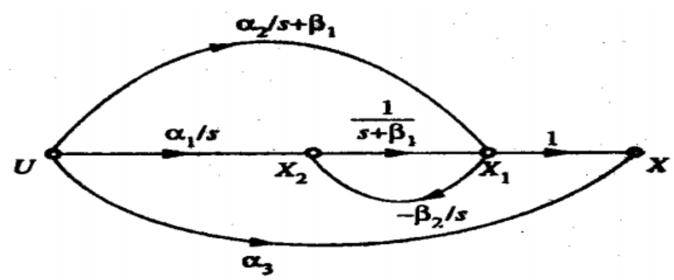
Example

For the system represented by the following equations, find the transfer function X(s)/U(s) by SFG technique.

$$x = x_1 + \alpha_3 u$$
$$\dot{x}_1 = -\beta_1 x_1 + x_2 + \alpha_2 u$$
$$\dot{x}_2 = -\beta_2 x_1 + \alpha_1 u$$

⇨ We need to Laplace transform the given sets of equations in order to represent differentiated variables.

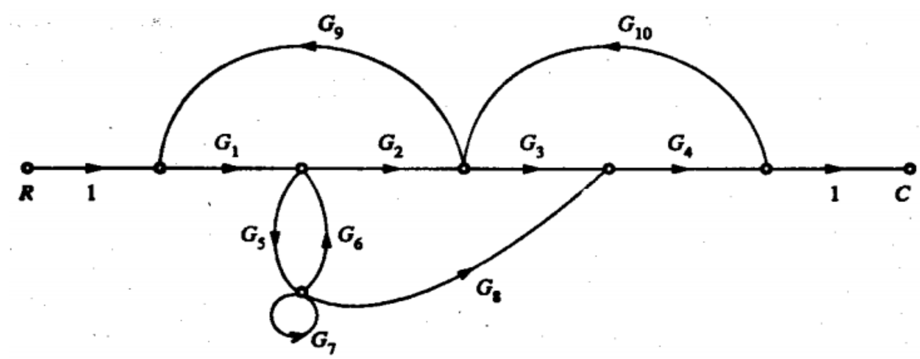
$$X = X_1 + \alpha_3 U$$
$$X_1 = \frac{1}{s+\beta_1} X_2 + \frac{\alpha_2}{s+\beta_1} u$$
$$X_2 = -\frac{\beta_2}{s} X_1 + \frac{\alpha_1}{s} u$$



$$\frac{X(s)}{U(s)} = \frac{\alpha_1 + \alpha_2 s + \alpha_3 \cdot [s^2 + \beta_1 s + \beta_2]}{s^2 + \beta_1 s + \beta_2} \qquad \{\text{Answer}\}$$

Example

Using Mason's gain formulae find C/R of the SFG shown in Figure below.

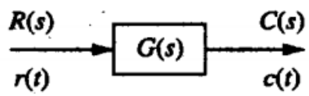


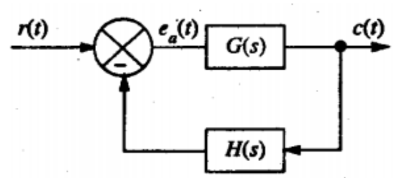
$$\frac{C}{R} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$$

$$= \frac{G_1 G_2 G_3 G_4 (1 - G_7) + G_1 G_5 G_8 G_4}{1 - [G_1 G_2 G_9 + G_3 G_4 G_{10} + G_1 G_5 G_8 G_4 G_{10} G_9 + G_5 G_6 + G_7] + [G_1 G_2 G_9 G_7 + G_3 G_4 G_{10} G_5 G_6 + G_3 G_4 G_{10} G_7]}$$

FEEDBACK CHARACTERISTICS OF CONTROL SYSTEMS

Consider the block diagram of the open-loop and the closed-loop system shown below.





For open-loop system, $C(s) = G(s)R(s)$

For closed-loop system, $C(s) = G(s)E_a(s) = G(s)[R(s) - H(s)C(s)]$

Hence, we have, $C(s) = \dfrac{G(s)}{1+G(s)H(s)}R(s)$ and, $E_a(s) = \dfrac{1}{1+G(s)H(s)}R(s)$

It is seen from the above equations that in order to reduce error, the loop-gain $G(s)H(s)$ should be made large over the range of frequencies of interest, i.e., $|G(s)H(s)|$? 1.

1. Reduction of parameter variations by use of feedback

One of the important properties of negative feedback systems is the reduction in the sensitivity to the variation in the parameters of the forward path. In the design of control systems, it is important that the transfer function of the closed-loop system be relatively insensitive to small changes in the values of the parameters of the components in the forward path of the system.

Let μ be a parameter of G(s). Then the sensitivity of G(s) with respect to the parameter μ is defined as,

$$S_\mu^G = \frac{\text{Fractional change in } G(s)}{\text{Fractional change in } \mu} = \frac{\Delta G / G}{\Delta \mu / \mu} = \frac{\mu}{G}\cdot\frac{\Delta G}{\Delta \mu} .$$

Now, $$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)};$$

$$S_\mu^T = \frac{\mu}{T}\cdot\frac{\Delta T}{\Delta \mu} = \frac{\mu}{G}\cdot\frac{G}{T}\cdot\frac{\Delta G}{\Delta \mu}\frac{\Delta T}{\Delta G} = S_\mu^G\cdot(1+GH)\cdot\frac{1+\cancel{GH}-\cancel{GH}}{(1+GH)^2} = \frac{S_\mu^G}{1+G(s)H(s)}$$

Thus feedback has reduced sensitivity in the variation in μ by the factor $\dfrac{1}{1+GH}$.

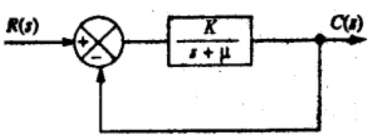
Again, $S_\mu^T = \frac{\mu}{T}\cdot\frac{\Delta T}{\Delta\mu} = \frac{\mu}{H}\cdot\frac{H}{T}\cdot\frac{\Delta H}{\Delta\mu}\cdot\frac{\Delta T}{\Delta H} = S_\mu^H\cdot\frac{H(1+GH)}{\cancel{G}}\cdot\frac{-G.\cancel{G}}{(1+GH)^2} = -S_\mu^H\cdot\frac{GH}{1+GH} \cong -S_\mu^H$.

It is seen that, the magnitude of two sensitivities are nearly equal for the variation of parameter in the feedback path. Thus, feedback does not reduce the sensitivity to variation in the parameter in feedback path.

Therefore, we can conclude that, G(s) in a closed-loop system may be less rigidly specified. On the other hand, we must be careful in accuracy of H(s) in the feedback loop.

2. Control over system dynamics by use of feedback

Let us consider the simple feedback system shown below.



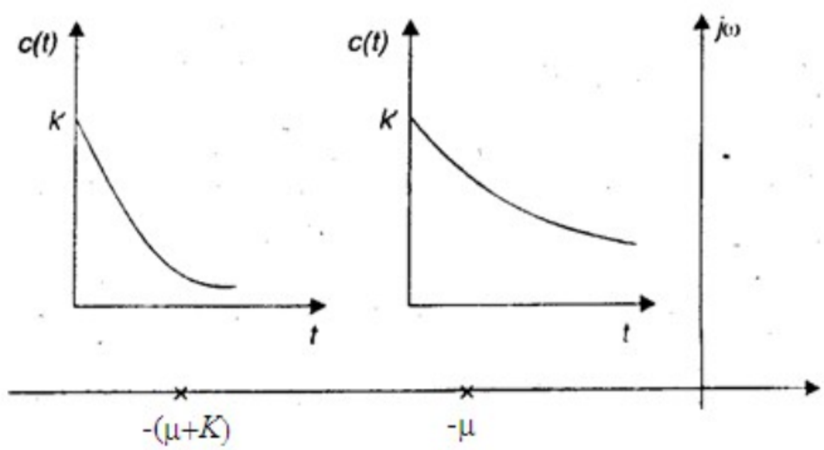
The open-loop transfer function is, $G(s) = \frac{K}{s+\mu}$.

The impulse response for the non-feedback system would be, $c(t) = Ke^{-\mu t}u(t) = Ke^{-t/\tau_1}u(t)$.

The closed-loop transfer function of the above system is, $T(s) = \frac{K}{s+\mu+K}$.

The impulse response of the closed-loop system is, $c(t) = Ke^{-(\mu+K)t}u(t) = Ke^{-t/\tau_2}u(t)$.

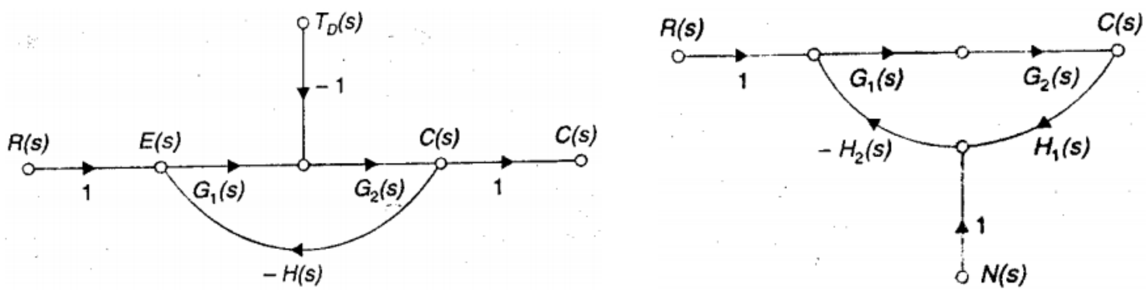
The location of the pole and the dynamic response of the non-feedback and feedback system are shown in Figure below.



It is seen that the time-constant of open-loop system is $\tau_1 = 1/\mu$ and that of closed-loop system is $\tau_2 = 1/(\mu + K)$. As the time-constant of closed-loop system is less, its dynamic response is faster than the same of the open-loop system.

3. Control of the effect of disturbance signal by use of feedback

A. Disturbance in the forward path



$$\frac{C_d(s)}{T_d(s)} = \frac{-G_2(s)}{1+G_1(s)G_2(s)H(s)} \cong \frac{-1}{G_1(s)H(s)}; \quad \text{or,} \quad C_d(s) = \frac{-T_d(s)}{G_1(s)H(s)}$$

If $G_1(s)$ is made very large, the effect of disturbance on the output will be very small.

B. Disturbance in the feedback path

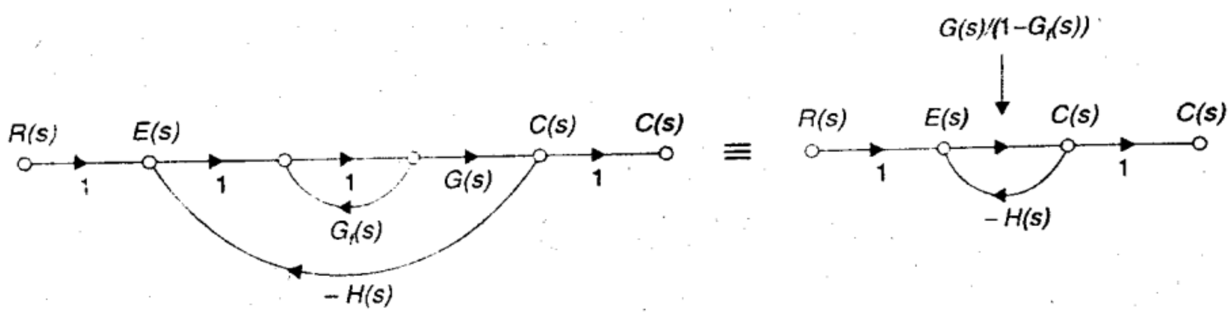
$$\frac{C_n(s)}{N(s)} = \frac{-G_1(s)G_2(s)H_2(s)}{1+G_1(s)G_2(s)H_1(s)H_2(s)} \cong \frac{-1}{H_1(s)}$$

Therefore, the effect of noise on output is, $C_n(s) \cong \frac{-1}{H_1(s)} \cdot N(s)$.

Thus, for the optimum performance of the system, the measurement sensor should be designed such that $H_1(s)$ is maximum. This is equivalent to maximizing the SNR of the sensor.

4. Regenerative Feedback

The regenerative feedback is sometimes used for increasing the loop gain of the feedback system. Figure in the following shows a feedback system where regenerative feedback occurs in the inner loop.



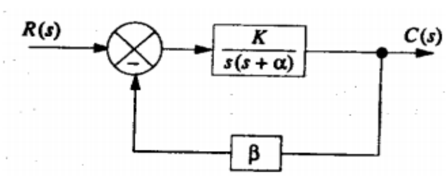
The open-loop gain is, $G_o(s) = \dfrac{G(s)}{1 - G_a(s)}$.

The system response is obtained as, $C(s) = \dfrac{R(s)\cdot G(s)/1 - G_a(s)}{1 + G_a(s)G(s)/1 - G_a(s)} = \dfrac{R(s)\cdot G(s)}{1 - G_a(s) + G(s)H(s)}$

When, $G_a(s)$; 1, $C(s) \cong \dfrac{R(s)}{H(s)}$. Due to high loop gain provided by the inner regenerative feedback loop, the closed-loop transfer function becomes insensitive to G(s).

Example

A position control system is shown below. Assume, K=10, $\alpha = 2$, $\beta = 1$. Evaluate: $S_K^T, S_\alpha^T, S_\beta^T$. For $r(t) = 2\cos 0.5t$ and a 5% change in K, evaluate the steady-state response and the change in steady-state response.



Here, $G(s) = \dfrac{K}{s(s+\alpha)}$, and $H(s) = \beta$

$$S_K^G = \frac{K}{G}\cdot\frac{dG}{dK} = s(s+\alpha)\cdot\frac{1}{s(s+\alpha)} = 1;$$

$$S_\alpha^G = \frac{\alpha}{G}\cdot\frac{dG}{d\alpha} = \frac{-\alpha}{s+\alpha} = \frac{-2}{s+2};\; S_\beta^H = \frac{\beta}{H}\cdot\frac{dH}{d\beta} = 1$$

$$S_K^T = \frac{S_K^G}{1+G(s)H(s)} = \frac{s(s+\alpha)}{s(s+\alpha)+K} = \frac{s^2+2s}{s^2+2s+10}$$

Therefore, $S_\alpha^T = \dfrac{S_\alpha^G}{1+G(s)H(s)} = \dfrac{-\alpha}{s+\alpha}\cdot\dfrac{s(s+\alpha)}{s(s+\alpha)+K} = \dfrac{-2s}{s^2+2s+10}$

$$S_\beta^T = \frac{-S_\beta^H\cdot G(s)H(s)}{1+G(s)H(s)} = \frac{-K}{s(s+\alpha)+K} = \frac{-10}{s^2+2s+10}$$

Now, $T(s) = \dfrac{K}{s^2+\alpha s+K\beta} = \dfrac{10}{s^2+2s+10}$; At $s = j0.5$, $T(j0.5) = 1.02e^{-j0.102}$

Thus, $c_{ss}(t) = 2.04\cos(0.5t - 0.102)$

Again, $S_K^T = \frac{K}{T}\cdot\frac{\Delta T}{\Delta K} \Rightarrow \frac{\Delta T}{T} = S_K^T \cdot \frac{\Delta K}{K} = \frac{s^2+2s}{s^2+2s+10}\cdot 0.05$

$\Rightarrow \Delta T(s) = \frac{s^2+2s}{s^2+2s+10}\times 0.05\times\frac{10}{s^2+2s+10} = \frac{0.5s(s+2)}{(s^2+2s+10)^2}$; $\Rightarrow \Delta T(j0.5) = 0.005e^{-j4.672}$

Thus, $\Delta c_{ss}(t) = \Delta T(j0.5)\times 2\cos 0.5t = 0.01\cos(0.5t - 4.672)$