BIOLOGICAL CONTROL SYSTEMS

ROUTH STABILITY CRITERION CONTINUED

Routh–Hurwitz criterion for second, third, and fourth-order polynomials[edit]

In the following, we assume the coefficient of the highest order (e.g. a_2 in a second order polynomial) to be positive. If necessary, this can always be achieved by multiplication of the polynomial with -1.

- For a second-order polynomial, $P(s) = a_2 s^2 + a_1 s + a_0 = 0$, all the roots are in the left half plane (and the system with characteristic equation $P(s)$ is stable) if all the coefficients satisfy $a_n > 0$.
- For a third-order polynomial $P(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$, all the coefficients must satisfy $a_n > 0$, and $a_2 a_1 > a_3 a_0$
- For a fourth-order polynomial $P(s) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$, all the coefficients must satisfy $a_n > 0$, and $a_3 a_2 > a_4 a_1$ and $a_3 a_2 a_1 > a_4 a_1^2 + a_3^2 a_0$
- In general Routh stability criterion proclaims that all First column elements of Routh array is to be of same sign.

This criterion is also known as modified Hurwitz Criterion of stability of the system. We will study this criterion in two parts. Part one will cover necessary condition for stability of the system and part two will cover the sufficient condition for the stability of the system. Let us again consider the characteristic equation of the system as

1) Part one (necessary condition for stability of the system): In this we have two conditions which are written below: (a) All the coefficients of the characteristic equation should be positive and real. (b) All the coefficients of the characteristic equation should be non zero.

2)Part two (sufficient condition for stability of the system): Let us first construct routh array. In order to construct the routh array follow these steps: (a) The first row will consist of all the even terms of the characteristic equation. Arrange them from first (even term) to last (even term). The first row is written below: a0 a2 a4 a6............ (b) The second row will consist of all the odd terms of the characteristic equation. Arrange them from first (odd term) to last (odd term). The first row is written below: a1 a3 a5 a7........... (c) The elements of third row can be calculated as:

(1) First element : Multiply a0 with the diagonally opposite element of next column (i.e. a3) then subtract this from the product of a1 and a2 (where a2 is diagonally opposite element of next column) and then finally divide the result so obtain with a1. Mathematically we write as first element

(2) Second element : Multiply a0 with the diagonally opposite element of next to next column (i.e. a5) then subtract this from the product of a1 and a4 (where a4 is diagonally opposite element of next to next column) and then finally divide the result so obtain with a1. Mathematically we write as second elemenT

Similarly, we can calculate all the elements of the third row. (d) The elements of fourth row can be calculated by using the following procedure: (1) First element : Multiply b1 with the diagonally opposite element of next column (i.e. a3) then subtract this from the product of a1 and b2 (where b2 is diagonally opposite element of next column) and then finally divide the result so obtain with b1. Mathematically we write as first element

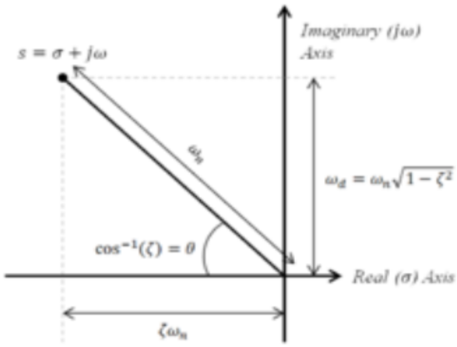
(2) Second element :Multiply b1 with the diagonally opposite element of next to next column (i.e. a5) then subtract this from the product of a1 and b3 (where b3 is diagonally opposite element of next to next column) and then finally divide the result so obtain with a1. Mathematically we write as second element

Similarly, we can calculate all the elements of the fourth row. Similarly, we can calculate all the elements of all the rows. Stability criteria if all the elements of the first column are positive then the system will be stable. However if anyone of them is negative the system will be unstable. Now there are some special cases related to Routh Stability Criteria which are discussed below: (1) Case one: If the first term in any row of the array is zero while the rest of the row has at least one non zero term. In this case we will assume a very small value (ε) which is tending to zero in place of zero. By replacing zero with (ε) we will calculate all the elements of the Routh array. After calculating all the elements we will apply the limit at each element containing (ε). On solving the limit at every element if we will get positive limiting value then we will say the given system is stable otherwise in all the other condition we will say the given system is not stable. (2) Case second : When all the elements of any row of the Routh array are zero. In this case we can say the system has the symptoms of marginal stability. Let us first understand the physical meaning of having all the elements zero of any row. The physical meaning is that there are symmetrically located roots of the characteristic equation in the s plane. Now in order to find out the stability in this case we will first find out auxiliary equation. Auxiliary equation can be formed by using the elements of the row just above the row of zeros in the Routh array. After finding the auxiliary equation we will differentiate the auxiliary equation to obtain elements of the zero row. If there is no sign change in the new routh array formed by using auxiliary equation, then in this we say the given system is limited stable. While in all the other cases we will say the given system is unstable

ROOT LOCUS:

In control theory and stability theory, **root locus analysis** is a graphical method for examining how the roots of a system change with variation of a certain system parameter, commonly a gain within a feedback system. This is a technique used as a stability criterion in the field of control systems developed by Walter R. Evans which can determinestability of the system. The root locus plots the poles of the closed loop transfer function in the complex S plane as a function of a gain parameter (see pole–zero plot).

Uses

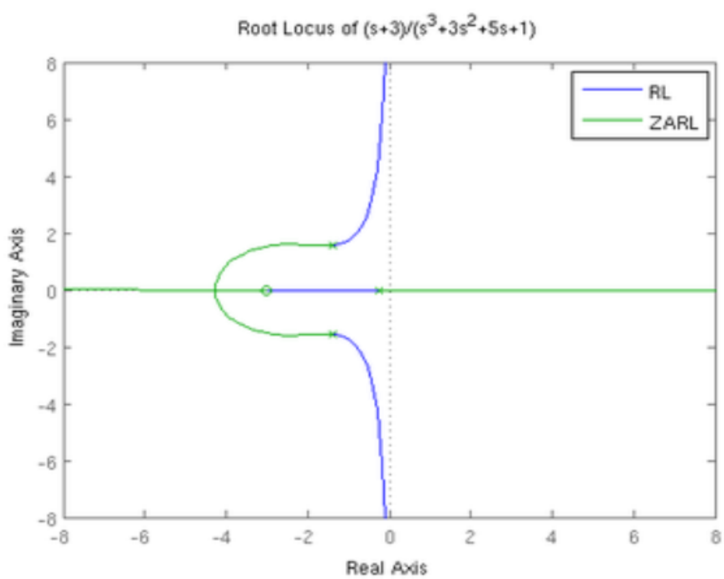


Effect of pole location on a second order system's natural frequency and damping ratio.

In addition to determining the stability of the system, the root locus can be used to design the damping ratio (ζ) and natural frequency (ω_n) of a feedback system. Lines of constant damping ratio can be drawn radially from the origin and lines of constant natural frequency can be drawn as arcs whose center points coincide with the origin. By selecting a point along the root locus that coincides with a desired damping ratio and natural frequency, a gain K can be calculated and implemented in the controller. More elaborate techniques of controller design using the root locus are available in most control textbooks: for instance, lag, lead, PI, PD and PID controllers can be designed approximately with this technique.

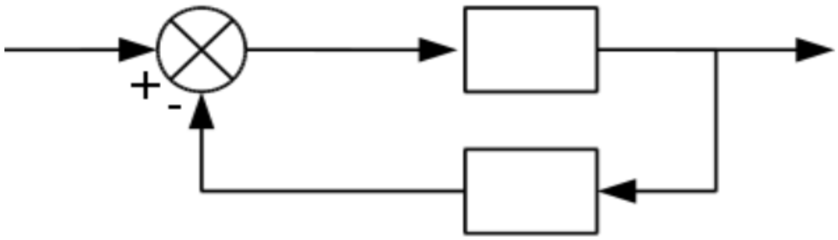
The definition of the damping ratio and natural frequency presumes that the overall feedback system is well approximated by a second order system; i.e. the system has a dominant pair of poles. This is often not the case, so it is good practice to simulate the final design to check if the project goals are satisfied.

Example



RL = root locus; ZARL = zero angle root locus

Suppose there is a feedback system whose input is the signal $X(s)$ and output is $Y(s)$. The feedback system forward path gain is $G(s)$; the feedback path gain is $H(s)$.



For this system, the overall transfer function is given by

$$T(s) = \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Thus the closed-loop poles (roots of the characteristic equation) of the transfer function are the solutions to the equation $1 + G(s)H(s) = 0$. The principal feature of this equation is that roots may be found wherever $G(s)H(s) = -1$.

In systems without pure delay, the product $G(s)H(s) = -1$ is a rational polynomial function and may be expressed as[2]

$$G(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_{m+n})}$$

where the $-z_i$ are the m zeros, the $-p_i$ are the $m + n$ poles, and K is a scalar gain. Typically, a root locus diagram will indicate the transfer function's pole locations for varying values of K. A root locus plot will be all those points in the s-plane where $G(s)H(s) = -1$ for any value of K.

The factoring of K and the use of simple monomials means the evaluation of the rational polynomial can be done with vector techniques that add or subtract angles and multiply or divide magnitudes. The vector formulation arises from the fact that each monomial term in the factored $G(s)H(s)$, $(s-a)$ for example, represents the vector from a to s. The polynomial can be evaluated by considering the magnitudes and angles of each of these vectors. According to vector mathematics, the angle of the result is the sum of all the angles in the numerator add minus the sum of all the angles in the denominator. Similarly, the magnitude of the result is the product of all the magnitudes in the numerator divided by the product of all the magnitudes in the denominator. It turns out that the calculation of the magnitude is not needed because K varies; one of its values may result in a root. So to test whether a point in the s-plane is on the root locus, only the angles to all the open loop poles and zeros need be considered. A graphical method that uses a special protractor called a "Spirule" was once used to determine angles and draw the root loci.

From the function $T(s)$, it can be seen that the value of K does not affect the location of the zeros. The root locus only gives the location of closed loop poles as the gain K is varied. The zeros of a system do not move.

Using a few basic rules, the root locus method can plot the overall shape of the path (locus) traversed by the roots as the value of K varies. The plot of the root locus then gives an idea of the stability and dynamics of this feedback system for different values of K.

Sketching root locus[edit]

- Mark open-loop poles and zeros
- Mark real axis portion to the left of an odd number of poles and zeros
- Find asymptotes

Let P be the number of poles and Z be the number of zeros:

$$P - Z = \text{number of asymptotes}$$

The asymptotes intersect the real axis at α (which is called the centroid) and depart at angle ϕ given by:

$$\phi_l = \frac{180^\circ + (l-1)360^\circ}{P - Z}, l = 1, 2, \ldots, P - Z$$

$$\alpha = \frac{\sum_P - \sum_Z}{P - Z}$$

where $\sum_P$ is the sum of all the locations of the poles, and $\sum_Z$ is the sum of all the locations of the explicit zeros.

- Phase condition on test point to find angle of departure
- Compute breakaway/break-in points

The breakaway points are located at the roots of the following equation:

$$\frac{dG(s)H(s)}{ds} = 0 \text{ or } \frac{d\overline{GH}(z)}{dz} = 0$$

Once you solve for z, the real roots give you the breakaway/reentry points. Complex roots correspond to a lack of breakaway/reentry.

z-plane versus s-plane

The root locus method can also be used for the analysis of sampled data systems by computing the root locus in the z-plane, the discrete counterpart of the s-plane. The equation $z = e^{sT}$ maps continuous s-plane poles (not zeros) into the z-domain, where T is the sampling period. The stable, left half s-plane maps into the interior of the unit circle of the z-plane, with the s-plane origin equating to $|z| = 1$ (because $e^0 = 1$). A diagonal line of constant damping in the s-plane maps around a spiral from (1,0) in the z plane as it curves in toward the origin. Note also that the Nyquist aliasing criteria is expressed graphically in the z-plane by the x-axis, where $\omega nT = \pi$.

The line of constant damping just described spirals in indefinitely but in sampled data systems, frequency content is aliased down to lower frequencies by integral multiples of the Nyquist frequency. That is, the sampled response appears as a lower frequency and better damped as well since the root in the *z*-plane maps equally well to the first loop of a different, better damped spiral curve of constant damping. Many other interesting and relevant mapping properties can be described, not least that *z*-plane controllers, having the property that they may be directly implemented from the *z*-plane transfer function (zero/pole ratio of polynomials), can be imagined graphically on a *z*-plane plot of the open loop transfer function, and immediately analyzed utilizing root locus.

Since root locus is a graphical angle technique, root locus rules work the same in the *z* and *s* planes.

The idea of a root locus can be applied to many systems where a single parameter *K* is varied. For example, it is useful to sweep any system parameter for which the exact value is uncertain in order to determine its behavior.