Arithmetization of Syntax

So far, we introduced the notion of representability and showed that many functions and relations are representable in *Cn* A_E.

By encoding the syntax of first-order logic using natural numbers, we can encode facts about the terms and formulas of logic as relations in $\mathcal{N}$.

We can then use our results about representability to show that there are some surprising limits to what can be represented in *Cn* A_E.

We begin by assigning a number to each symbol in our formal language.

Symbol	Encoded symbol	Symbol	Encoded symbol
$\forall$	0	$($	1
$\mathbf{0}$	2	$)$	3
$\mathbf{S}$	4	$\neg$	5
$<$	6	$\rightarrow$	7
$+$	8	$=$	9
$\times$	10	v_1	11
E	12	v_2	13
		v_k	9+2k

Note that this encoding could be modified to accommodate any countable signature (the symbols in the signature are assigned to the even numbers).

Let h be the function which maps each symbol to its encoding.

Gödel Numbers

For an expression $\epsilon = s_0 \cdots s_n$ of the language, we define its Gödel number, $\#(\epsilon)$ by

$$\#(s_0, \cdots, s_n) = \langle h(s_0), \ldots, h(s_n) \rangle.$$

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Example

$$\begin{aligned} \#(\exists\, v_3\, v_3 = \mathbf{0}) &= \#((\neg \forall\, v_3\, (\neg = v_3 \mathbf{0}))) \\ &= \langle 1, 5, 0, 15, 1, 5, 9, 15, 2, 3, 3 \rangle \\ &= 2^2 \cdot 3^6 \cdot 5^1 \cdot 7^{16} \cdot 11^2 \cdot 13^6 \cdot 17^{10} \cdot 19^{16} \cdot 23^3 \cdot 29^4 \cdot 31^4. \end{aligned}$$

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A set of expressions corresponds to a set of Gödel numbers:

$$\#\Phi = \{\#(\epsilon) | \epsilon \in \Phi\}.$$

We now show that various relations and functions involving Gödel numbers are representable.

1. The set of Gödel numbers of variables is representable.

Proof

A formula which defines this set is:

$$\exists\, b\, (b < a \wedge a = \langle 11 + 2 \cdot b \rangle \}.$$

This formula makes use of bounded quantification, the equality relation, arithmetic constants, sequence numbers, and function composition, all of which we showed to be representable earlier.

2. The set of Gödel numbers of terms is representable.

Proof idea

Let f be the corresponding characteristic function (i.e. the function whose value is 1 if its input is the Gödel number of a term and 0 otherwise).

Then,

$$f(a) = \begin{cases} 1 & \text{if } a \text{ is the Gödel number of a variable,} \\ 1 & \text{if } \exists\, i < a^{a \cdot lh(a)}, \exists\, k < a \\ & [i \text{ is a sequence number and} \\ & \forall\, j < lh(i)\ (f((i)_j) = 1 \text{ and} \\ & k \text{ is the value of } h \text{ at some } (lh(i))\text{-place function symbol and} \\ & a = \langle k \rangle * *_{j < lh(i)} (i)_j] \\ 0 & \text{otherwise.} \end{cases}$$

3. The set of Gödel numbers of atomic formulas is representable.

4. The set of Gödel numbers of wffs is representable.

5. There is a representable function Sb such that for a term or formula α, variable x, and term t,

$$Sb\,(\#\alpha, \#x, \#t) = \#(\alpha^x_t).$$

6. The function whose value at n is $\#(\mathbf{S}^n\mathbf{0})$ is representable.

7. There is a representable relation Fr such that for a term or formula α and a variable x, $\langle \#\alpha, \#x \rangle \in Fr$ iff x occurs free in α.

8. The set of Gödel numbers of sentences is representable.

9. There is a representable relation Sbl such that for a formula α, variable x, and term t, $\langle \#a, \#x, \#t \rangle \in Sbl$ iff t is substitutable for x in α.

10. The relation Gen, where $\langle a, b \rangle \in Gen$ iff a is the Gödel number of a formula and b is the Gödel number of a generalization of that formula, is representable.

11. The set of Gödel numbers of tautologies is representable.

⋮

17. The set of Gödel numbers of logical axioms is representable.

Let $\mathcal{G}(\langle \alpha_0, \ldots, \alpha_n \rangle) = \langle \#\alpha_0, \ldots, \#\alpha_n \rangle$.

18. For a finite set A of formulas,

$$\{\mathcal{G}(D) | D \text{ is a deduction from } A\}.$$

is representable.

19. Any recursive relation is representable in $Cn\,A_E$.

⋮

21. If $\#A$ is recursive and $Cn\,A$ is a complete theory, then $\#Cn\,A$ is recursive.

Church's Thesis (Again)

Using techniques like those we have just seen (representability using Gödel numbers), it can be shown that the function computed by any model of computation is recursive.

This is the reason we feel justified in accepting Church's thesis which states that a relation is decidable iff the relation is recursive.

We have just shown that every recursive relation is representable in the theory *Cn* A_E. This means that *Cn* A_E is powerful enough to represent any decidable relation.

Fixed-Point Lemma

Suppose β is a formula which defines (in N) some subset A of $\mathcal{N}$.

How do we interpret the following formulas:

- $\beta(\mathbf{S}^n\mathbf{0})$

  $n \in A$

- $\beta(\mathbf{S}^{\#\sigma}\mathbf{0})$

  $\#\sigma \in A$

- $\sigma \leftrightarrow \beta(\mathbf{S}^{\#\sigma}\mathbf{0})$

  $\models_N \sigma$ iff $\#\sigma \in A$

The fixed-point lemma gives us the surprising result that for *any* such formula β, we can always find a sentence σ such that the last formula not only is true in N, but is derivable from A_E.

Theorem

Given any formula β in which only v_1 occurs free, we can find a sentence σ such that $A_E \vdash \left[\sigma \leftrightarrow \beta(\mathbf{S}^{\#\sigma}\mathbf{0})\right]$.

Proof

Let $\theta(v_1, v_2)$ functionally represent in $Cn\, A_E$ a function h whose value at $\#\alpha$ is $\#(\alpha(\mathbf{S}^{\#\alpha}\mathbf{0}))$:

Let $Sb\,(\#\alpha, \#x, \#t) = \#(\alpha^x_t)$.
Let $f(n) = \#(\mathbf{S}^n\mathbf{0})$.
Let $g(\#\alpha, n) = Sb\,(\#\alpha, \#v_1, f(n)) = \#\alpha(\mathbf{S}^n\mathbf{0})$.
Let $h(\#\alpha) = g(\#\alpha, \#\alpha) = \#\alpha(\mathbf{S}^{\#\alpha}\mathbf{0})$.

Consider the formula $\gamma \equiv \forall\, v_2\ \ [\theta(v_1, v_2) \rightarrow \beta(v_2)]$.

This defines in N a set to which $\#\alpha$ belongs iff $h(\#\alpha)$ is in the set defined by β.

Now, let $\sigma \equiv \gamma(\mathbf{S}^{\#\gamma}\mathbf{0})$.

We have $h(\#\alpha) = \#(\alpha(\mathbf{S}^{\#\alpha}\mathbf{0}))$,
γ defines some set Γ such that $\#\alpha \in \Gamma$ iff $h(\#\alpha)$ is in the set defined by β,
and $\sigma \equiv \gamma(\mathbf{S}^{\#\gamma}\mathbf{0})$.

σ holds in N	iff	$\gamma(\mathbf{S}^{\#\gamma}\mathbf{0})$ holds in N
	iff	$\#\gamma \in \Gamma$
	iff	$h(\#\gamma)$ is in the set defined by β
	iff	$\#\gamma(\mathbf{S}^{\#\gamma}\mathbf{0})$ is in the set defined by β
	iff	$\#\sigma$ is in the set defined by β
	iff	$\beta(\mathbf{S}^{\#\sigma}\mathbf{0})$ is true in N.

This shows that σ holds in N iff $\beta(\mathbf{S}^{\#\sigma}\mathbf{0})$ holds in N, i.e.

$$\models_N \left[\sigma \leftrightarrow \beta(\mathbf{S}^{\#\sigma}\mathbf{0})\right].$$

However, we need to show that this fact is deducible from A_E:

$$A_E \vdash \left[\sigma \leftrightarrow \beta(\mathbf{S}^{\#\sigma}\mathbf{0})\right].$$

The proof that this is derivable follows from the fact that θ functionally represents h and the definition of σ. $\square$

Tarski Undefinability Theorem

Theorem

The set $\#\mathit{Th}\,N$ is not definable in N.

Proof

Suppose that β were a formula which defined the set $\#\mathit{Th}\,N$. Applying the fixed-point lemma to $\neg\beta$, we get a sentence σ such that

$$\models_N \left[\sigma \leftrightarrow \neg\beta(\mathbf{S}^{\#\sigma}\mathbf{0})\right].$$

and thus,

$$\models_N \sigma \text{ iff } \not\models_N \beta(\mathbf{S}^{\#\sigma}\mathbf{0}).$$

So, if $\sigma \in \mathit{Th}\,N$ ($\models_N \sigma$), then its Gödel number is not in the set β defines, meaning that β cannot define $\#\mathit{Th}\,N$.

On the other hand, if $\sigma \notin \mathit{Th}\,N$ ($\not\models_N \sigma$), then its Gödel number *is* in the set β defines, meaning that β cannot define $\#\mathit{Th}\,N$. $\square$

Corollary

The set $\#\mathit{Th}\,N$ is not recursive.

Proof

Any recursive set is definable in N.

In other words, $\#\mathit{Th}\,N$ (and thus $\mathit{Th}\,N$) is not decidable.

What happened?

Essentially, what we have shown is that the language and structure of N are powerful enough that for any decidable set D, we can always find a sentence whose meaning in N is “I am in (or not in) D”.

Thus, if $\mathit{Th}\,N$ were decidable, there would be a sentence whose meaning in N is “I am not in $\mathit{Th}\,N$”. If the sentence is in $\mathit{Th}\,N$, then it’s not, and if it’s not, then it is. Contradiction.

Theorem

If $A \subseteq \mathit{Th}\, N$ and $\#A$ is recursive, then $\mathit{Cn}\, A$ is not a complete theory.

Proof

Suppose $\mathit{Cn}\, A$ is complete. Since $A \subseteq \mathit{Th}\, N$, it follows that $\mathit{Cn}\, A \subseteq \mathit{Th}\, N$. And since $\mathit{Cn}\, A$ is complete and $\mathit{Th}\, N$ is satisfiable, we must have $\mathit{Cn}\, A = \mathit{Th}\, N$. But then $\#\mathit{Cn}\, A = \#\mathit{Th}\, N$ is recursive (by item 21). But every recursive set is definable in N, and we just showed that $\#\mathit{Th}\, N$ is not definable in N.

Hence, $\mathit{Th}\, N$ cannot be axiomatized.

Strong Undecidability of $\mathit{Cn}\, A_E$

Theorem

Let T be any theory (in the language of N) such that $T \cup A_E$ is consistent. Then $\#T$ is not recursive.

Proof

Let T' be the theory $\mathit{Cn}\,(T \cup A_E)$. If $\#T$ is recursive, then it is not hard to show given our results on representability that $\#T'$ is also recursive.

This means that there is a formula β which represents $\#T'$ in $\mathit{Cn}\, A_E$. By the fixed-point lemma, there is a sentence σ such that

$$A_E \vdash \left[\sigma \leftrightarrow \neg\beta(\mathbf{S}^{\#\sigma}\mathbf{0})\right].$$

Essentially, σ says "I am not in T'". As before, this leads to a contradiction.

Corollary: Alternate Version of Incompleteness Theorem

Assume that $\#\Sigma$ is recursive and $\Sigma \cup A_E$ is consistent. Then $\mathit{Cn}\, \Sigma$ is not complete.

General Undecidability Results

The previous theorem has significant consequences in terms of what is decidable in first-order logic.

Church's Theorem

The set of Gödel numbers of valid sentences (in the language of N) is not recursive.

Proof

In the strong undecidability theorem, take T to be the set of valid sentences.

Thus, the language of N is an example of a first-order language whose valid sentences cannot be decided by any effective procedure.

In fact, it is known that a single two-place predicate symbol is sufficient for the set of valid formulas of a language to be undecidable.

Second Incompleteness Theorem

Suppose A is a recursive set of sentences and $T = \mathit{Cn}\, A$. Then,

$$a \in \#T \quad \text{iff} \quad \exists\, d\, [\; d \text{ is the number of a deduction from } A \text{ and the last component of } d \text{ is } a \text{ and } a \text{ is the Gödel number of a sentence} \;].$$

Is the set $\#T$ recursive?

Most of the time it is not. For example, if A is consistent with A_E, then by strong undecidability, $\#T$ is not recursive.

Is the set of pairs $\langle a, d \rangle$ satisfying the condition in brackets recursive?

Yes. It is built out of parts which we showed to be representable in $\mathit{Cn}\, A_E$. A relation is recursive iff it is representable in $\mathit{Cn}\, A_E$.

Suppose A is a recursive set of sentences and $T = \mathit{Cn}\, A$. Then,

$a \in \#T$ iff $\exists\, d\, [$ d is the number of a deduction from A and the last component of d is a and a is the Gödel number of a sentence $]$.

Let $\pi(v_1, v_2)$ be a formula which represents (in A_E) the set of pairs $\langle a, d \rangle$ satisfying the condition in brackets.

If σ is a sentence, then what is the meaning of the sentence

$\exists\, v_2\; \pi(\mathbf{S}^{\#\sigma}\mathbf{0}, v_2)$?

$\exists\, v_2\; \pi(\mathbf{S}^{\#\sigma}\mathbf{0}, v_2)$ iff $\sigma \in T$ (or, equivalently, $A \vdash \sigma$ or $T \vdash \sigma$).

Define $Prb_T\sigma = \exists\, v_2\; \pi(\mathbf{S}^{\#\sigma}\mathbf{0}, v_2)$.

Suppose T contains A_E. What is the meaning of the sentence $\neg Prb_T \mathbf{0} = \mathbf{S0}$?

$\neg Prb_T \mathbf{0} = \mathbf{S0}$ iff T is consistent. Define $Cons\, T = \neg Prb_T \mathbf{0} = \mathbf{S0}$.

Lemma

Let $T = \mathit{Cn}\, A$, where A is recursive.

1. Whenever $T \vdash \sigma$, then $A_E \vdash Prb_T\sigma$.
2. If $A_E \subseteq T$, then T has the "reflection" property:

   $T \vdash \sigma$ implies $T \vdash Prb_T\sigma$

Proof

(1) Suppose $T \vdash \sigma$. Then there is some deduction of σ from A. Let d be the Gödel number of this deduction. Then $A_E \vdash \pi(\mathbf{S}^{\#\sigma}\mathbf{0}, \mathbf{S}^{d}\mathbf{0})$, and hence $A_E \vdash Prb_T\sigma$. (2) Follows immediately from (1).

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Thus, whenever such a theory T can prove a sentence σ, it can prove that it can prove σ.

We have $Prb_T\sigma = \exists\, v_2\, \pi(\mathbf{S}^{\#\sigma}\mathbf{0}, v_2)$.

Let $\beta = \exists\, v_2\, \pi(v_1, v_2)$. By the fixed-point lemma, there is a sentence σ such that

$A_E \vdash (\sigma \leftrightarrow \neg\beta(\mathbf{S}^{\#\sigma}\mathbf{0}))$, or
$A_E \vdash (\sigma \leftrightarrow \neg Prb_T\sigma)$.

Lemma

Let $T = Cn\, A$, where A is recursive, and suppose that $A_E \subseteq T$. Let σ be obtained from the fixed-point lemma as above. If T is consistent, then $T \nvdash \sigma$.

Proof

We will prove the contrapositive.

If $T \vdash \sigma$, then $T \vdash Prb_T\sigma$ by reflection. But then, since $A_E \vdash (\sigma \leftrightarrow \neg Prb_T\sigma)$, it follows that $T \vdash \neg\sigma$, and so T is inconsistent.

We showed earlier that it is possible to formulate a sentence $Cons\,T$ which is true (in N) iff T is consistent. An interesting question is whether $Cons\,T \in T$. In other words, can T prove its own consistency?

We will show that under fairly modest assumptions, the answer is no. In other words, $Cons\,T$ is an example of a sentence which is either true but unprovable or provable but false.

To show this, we must introduce a couple of additional assumptions about T. We say that T is *sufficiently strong* if it has the following three properties.

1. $A_E \subseteq T$ (and thus we have the reflection property, $T \vdash \sigma$ implies $T \vdash Prb_T\sigma$).
2. For any sentence σ, $T \vdash (Prb_T\sigma \rightarrow Prb_T Prb_T\sigma)$ (formalized reflection).
3. For any sentences ρ and σ, $T \vdash (Prb_T(\rho \rightarrow \sigma) \rightarrow (Prb_T\rho \rightarrow Prb_t\sigma))$ (formalized modus ponens).

Lemma

Assume that $T = Cn\,A$ is sufficiently strong, and let σ be obtained from the fixed-point lemma so that $A_E \vdash (\sigma \leftrightarrow \neg Prb_T\sigma)$. Then $T \vdash (Cons\,T \rightarrow \neg Prb_T\sigma)$.

Proof

First note that because of the way we chose σ, we have:

$$T \vdash (\sigma \rightarrow (Prb_T\sigma \rightarrow \mathbf{0} = \mathbf{S0})).$$

By reflection and then formalized modus ponens, we get

$$T \vdash (Prb_T\sigma \rightarrow Prb_T(Prb_T\sigma \rightarrow \mathbf{0} = \mathbf{S0})).$$

Applying formalized modus ponens again, we get

$$T \vdash (Prb_T\sigma \rightarrow (Prb_T Prb_T\sigma \rightarrow \neg Cons\,T)).$$

Finally, using formalized reflection and a tautology, we get

$$T \vdash Prb_T\sigma \rightarrow \neg Cons\,T.$$

Theorem

If T is a sufficiently strong recursively axiomatizable theory, then $T \vdash Cons\, T$ iff T is inconsistent.

Proof

The "if" direction is trivial. Suppose $T \vdash Cons\, T$. By the previous lemma, it follows that $T \vdash \neg Prb_T \sigma$, and thus, by the choice of σ, $T \vdash \sigma$. It follows from our other lemma that T is inconsistent.

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So, sufficiently strong theories cannot prove their own consistency, but what theories are sufficiently strong?

One example is "Peano Arithmetic" (PA) which includes A_E as well as the "induction axioms": $\phi(0) \wedge \forall\, x\ (\phi(x) \to \phi(\mathbf{S}x)) \to \forall\, x\ \phi(x)$.

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However, the most important example of a sufficiently strong theory is axiomatic set theory.

This has important philosophical ramifications. It means that there is no way to prove, using the formalisms of mathematics (which are based on set theory) that set theory itself is consistent, providing a disappointing answer to Hilbert’s second problem.