

Mathematics for Science
Lecture 1
Arithmetic of Complex Numbers
Lecturer: Kahenya, N.P

Introduction to lecture 1

This lecture will introduce the set and arithmetic of complex numbers.

Intended learning outcomes

At the end of this lecture you will be able to;

- (i) Define a complex number.
- (ii) Operate the arithmetic of complex numbers.

References

These lecture notes should be complemented with relevant topics in (Kahenya, 2017; Stewart, 2012)

Complex numbers \mathbb{C}

A complex number \mathbb{C} is the set of numbers z that form an ordered pair (a, b) of real numbers such that $z = a + bi$ where $a = \text{Re}(z)$ i.e. the Real part of z and $b = \text{Im}(z)$ i.e. the imaginary part of z .

That is;

$$z = (a + bi) \in \mathbb{C} \text{ for some } a, b \in \mathbb{R}$$

Note that i is called the imaginary number where i is the root of negative one, that is, $i = \sqrt{-1}$

therefore $(i \times i) = i^2 = -1$.

For example; $2 + 3i$; $7i$, $9 - 5i$; $\sqrt{-10} = i\sqrt{10}$

Complex arithmetic

Definition 1(Sum and difference of complex numbers): When adding (or subtracting) two or more complex numbers, you need to add(subtract) the real parts together and imaginary parts together.

In general given the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + y_2i$ then

- (i) $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + (y_1i + y_2i) = (x_1 + x_2) + (y_1 + y_2)i$.
- (ii) $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + (y_1i - y_2i) = (x_1 - x_2) + (y_1 - y_2)i$.

Example: Evaluate; $(7 - 2i) + (3 + 5i)$

Solution: $(7 - 2i) + (3 + 5i) = (7 + 3) + (-2i + 5i) = 10 + 3i$

Example: Evaluate: $(6 + i) - (7 - 9i)$

Solution: $(6 + i) - (7 - 9i) = (6 - 7) + (i - -9i) = -1 + 10i$

Definition 2 (Product of Complex numbers): Given the complex numbers $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$ then;

$$\begin{aligned}z_1 z_2 &= (a_1 + b_1i)(a_2 + b_2i) = a_1 a_2 + a_1 b_2i + a_2 b_1i + (b_1i)(b_2i) \\ &= a_1 a_2 + (a_1 b_2 + a_2 b_1)i + b_1 b_2 i^2 \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i \text{ since } i^2 = -1\end{aligned}$$

Example: Evaluate: $(4 + 3i)(9 - 2i)$

Solution: $(4 + 3i)(9 - 2i) = 36 - 8i + 27i - 6i^2 = 36 + 19i + 6 = 42 + 19i$

Example: Now consider the following: $(x + yi)(x - yi)$

$$\Rightarrow x^2 - xyi + xyi - y^2 i^2 = x^2 + y^2$$

Note that the product of the two is a real number $(x^2 + y^2)$.

Definition 4 (Conjugate of complex number): Given $z = x + yi$ then $\bar{z} = x - yi$ is called the complex conjugate of $z = x + yi$ and vice versa.

Hence we have that, the product of a complex number and its conjugate is a real number.

Also the conjugate of a proper complex is itself. For example the conjugate of $7i$ is $7i$. In general, the complex conjugate of xi is xi since $xi \times xi = x^2 i^2 = -x^2 \in \mathbb{R}$

Example: Work out; $8i \times 8i = 64i^2 = -64$

Definition 5 (Division of complex numbers): Given the complex number $\frac{a+bi}{x+yi}$ we can simplify by multiplying both the numerator and denominator with the conjugate of the denominator i.e.

$$\frac{a + bi}{x + yi} \times \frac{x - yi}{x - yi} = \frac{(a - bi)(x - yi)}{x^2 + y^2}$$

Example: Simplify the following; $\frac{5+7i}{3+2i}$

Solution: We multiply both the numerator and the denominator by the conjugate of the denominator i.e. $3 - 2i$ to get;

$$\frac{5 + 7i}{3 + 2i} \times \frac{3 - 2i}{3 - 2i} = \frac{15 - 10i + 21i - 14i^2}{3^2 + 2^2} = \frac{29 + 11i}{13}$$

Definition 6: (Equal complex numbers) If two complex numbers are equal then their real parts and their imaginary parts are equal. That is if $x + yi = a + bi$ then $x = a$ and $y = b$.

Example: Find the value of a and b given that;

$$(2 + i)^2 - 3(a - bi) = a + bi$$

Solution: Consider the LHS:

$$\begin{aligned} &4 + 4i + i^2 - 3a + 3bi \\ &= 4 - 1 - 3a + 4i + 3bi \\ &= (3 - 3a) + (4 + 3b)i\end{aligned}$$

Then we have;

$$(3 - 3a) + (4 + 3b)i = a + bi$$

This implies that;

$$\begin{aligned} 3 - 3a &= a \therefore a = \frac{3}{4} \\ (4 + 3b)i &= bi \\ \Rightarrow 4 + 3b &= b \therefore b = -2 \end{aligned}$$

Example: Determine the value of a and b given;

$$\frac{a}{3+i} + \frac{b}{1-3i} = 7 + 5i$$

Solution: Consider the LHS. We simplify it to get;

$$\begin{aligned} \frac{a}{3+i} &= \frac{a}{3+i} \times \frac{3-i}{3-i} = \frac{a(3-i)}{3^2+1^2} = \frac{a(3-i)}{10} \\ \frac{b}{1-3i} &= \frac{b}{1-3i} \times \frac{1+3i}{1+3i} = \frac{b(1+3i)}{1^2+3^2} = \frac{b(1+3i)}{10} \end{aligned}$$

Hence we have;

$$\begin{aligned} \frac{a(3-i)}{10} + \frac{b(1+3i)}{10} &= \frac{3a - ai + b + 3bi}{10} = \frac{(3a+b) + (3b-a)i}{10} \\ &= \frac{3a+b}{10} + \frac{3b-a}{10}i \end{aligned}$$

Equating to the RHS we get;

$$\begin{aligned} \frac{3a+b}{10} + \frac{3b-a}{10}i &= 7 + 5i \\ \Rightarrow \frac{3a+b}{10} = 7 &\Rightarrow 3a+b = 70 \dots (i) \end{aligned}$$

Again;

$$\begin{aligned} \frac{3b-a}{10}i &= 5i \\ (3b-a)i &= 50i \\ \Rightarrow 3b-a &= 50 \dots (ii) \end{aligned}$$

We next solve equations (i) and (ii) simultaneously;

From (ii) $a = 3b - 50$

Replacing a with $3b - 50$ in equation (i) to get;

$$\begin{aligned} 3(3b-50) + b &= 70 \\ 9b - 150 + b &= 70 \\ 10b &= 220 \\ b &= 22 \end{aligned}$$

$$\therefore a = 3(22) - 50 = 66 - 50 = 16$$

Example: Find the value of a and b given;

$$\frac{a}{3-i} + \frac{bi}{i+5} = \frac{4}{2+i}$$

Solution: Consider the LHS:

$$\begin{aligned} \frac{a}{3-i} &= \frac{a}{3-i} \times \frac{3+i}{3+i} = \frac{a(3+i)}{3^2+1^2} = \frac{a(3+i)}{10} \\ \frac{bi}{i+5} &= \frac{bi}{i+5} \times \frac{i-5}{i-5} = \frac{bi(i-5)}{1^2+5^2} = \frac{-b-5bi}{26} \end{aligned}$$

Therefore we have;

$$\frac{a(3+i)}{10} + \frac{-b-5bi}{26} = \frac{26a(3+i) - 10b - 50bi}{260} = \frac{78a + 26ai - 10b - 50bi}{260}$$

$$\frac{(78a - 10b) + (26a - 50b)i}{260} = \frac{1}{260}(78a - 10b) + \frac{1}{260}(26a - 50b)i \dots (*)$$

Next we consider the RHS;

$$\frac{4}{2+i} = \frac{4}{2+i} \times \frac{2-i}{2-i} = \frac{4(2-i)}{2^2 + i^2} = \frac{8-4i}{5} = \frac{8}{5} - \frac{4}{5}i \dots (**)$$

Comparing the LHS and RHS i.e. (*) and (**) we get;

$$\frac{1}{260}(78a - 10b) + \frac{1}{260}(26a - 50b)i = \frac{8}{5} - \frac{4}{5}i$$

$$\Rightarrow \frac{1}{260}(78a - 10b) = \frac{8}{5} \dots (i)$$

$$\frac{1}{260}(26a - 50b) = -\frac{4}{5} \dots (ii)$$

From equation (i) we have;

$$78a - 50b = 416 \dots (iii)$$

From equation (ii) we have;

$$26a - 50b = -208 \dots (iv)$$

Subtracting equation (iv) from equation (iii) to get;

$$78a - 50b = 416$$

$$\underline{26a - 50b = -208}$$

$$52a = 624$$

Therefore $a = 12$

Replacing a with 12 in equation (iii) to get;

$$78(12) - 50b = 416$$

$$936 - 416 = 50b$$

$$520 = 50b$$

$$\therefore b = 10.4$$

Definition 7: (Square root of complex numbers)

Proposition: The square root of a complex numbers is a complex number i.e. $\sqrt{x + yi} = a + bi$

Example: Evaluate $\sqrt{5 + 3i}$

Solution: Let $\sqrt{5 + 3i} = x + yi$

Squaring both sides give us;

$$5 + 3i = (x + yi)^2$$

$$5 + 3i = x^2 + 2xyi + y^2i^2$$

$$5 + 3i = x^2 + 2xyi - y^2$$

$$5 + 3i = (x^2 - y^2) + 2xyi$$

Comparing the two sides we get two equations;

$$x^2 - y^2 = 5 \dots (i)$$

$$2xy = 3 \dots (ii)$$

From equation (ii) we have $y = \frac{3}{2x}$. Then replace y with $\frac{3}{2x}$ in equation (i) to get;

$$x^2 - \left(\frac{3}{2x}\right)^2 = 5$$

$$x^2 - \frac{9}{4x^2} = 5$$

$$4x^4 - 9 = 5x^2$$

$$4x^4 - 20x^2 - 9 = 0 \dots (*)$$

Let $t = x^2$ then equation (*) becomes $4t^2 - 20t - 9 = 0$

$$t = \frac{20 \pm \sqrt{400 + 4(4)(9)}}{8} = \frac{20 \pm \sqrt{544}}{8} \approx 5.42 \text{ or } -0.42$$

Hence;

$$x^2 = 5.42 \therefore x = \pm 2.33$$

$$x^2 = -0.42 \text{ (Since } x \text{ must be a real number then this is invalid)}$$

Hence ; when $x = \pm 2.33$ then $y = \frac{3}{2x} = \pm 0.64$

Therefore $\sqrt{5 + 3i} = x + yi \approx \pm(2.33 + 0.64i)$

Example: Evaluate; $\sqrt{1 - 5i}$

Solution: Let $\sqrt{1 - 5i} = x + yi$

Square both sides to get;

$$1 - 5i = (x + yi)^2 = x^2 + 2xyi - y^2$$

$$1 - 5i = (x^2 - y^2) + 2xyi \dots (*)$$

From equation (*) we can get two equations;

$$1 = x^2 - y^2 \dots (i)$$

$$-5 = 2xy \dots (ii)$$

From equation (ii) we get $y = -\frac{5}{2x}$ then we replace y with $-\frac{5}{2x}$ in equation (i) to get;

$$1 = x^2 - \left(-\frac{5}{2x}\right)^2$$

$$1 = x^2 - \frac{25}{4x^2}$$

$$4x^2 = 4x^4 - 25 \Rightarrow 4x^4 - 4x^2 - 25 = 0$$

Let $t = x^2$ then we have;

$$4t^2 - 4t - 25 = 0$$

$$t = \frac{4 \pm \sqrt{16 + 400}}{8} = \frac{4 \pm 20.4}{8} = \frac{24.4}{8} \text{ or } -\frac{16.4}{8}$$

Therefore

$$x^2 = \frac{24.4}{8} \Rightarrow x = \pm 1.75 \therefore y \approx \pm 1.429$$

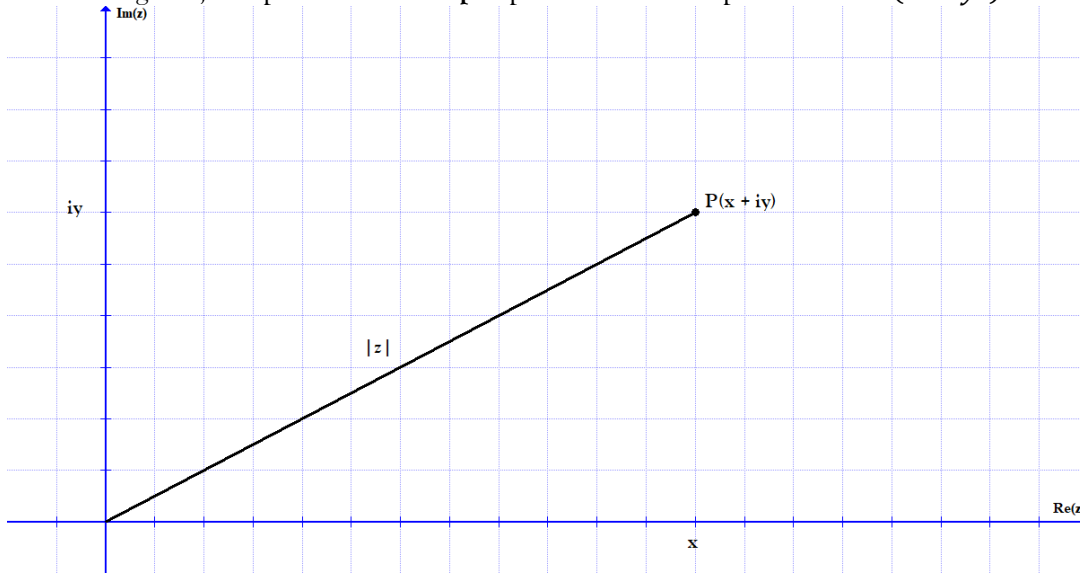
$$x^2 = -\frac{16.4}{8} \text{ (Invalid since } x \text{ must be a Real)}$$

Hence;

$$\sqrt{1 - 5i} = \pm(1.75 + 1.43i)$$

Definition 8: (Argand diagram)

A complex number can be represented as a vector in the Argand diagram. If point $P(x, y)$ is a point on the diagram, the position vector \mathbf{p} represents the complex number $(x + yi)$.



Definition 9: (Modulus of a complex number)

The distance \overline{OP} is called the modulus of z and is denoted by r , $\text{mod } z$, $|z|$, or $|x + iy|$. It is the measure of magnitude z . By Pythagoras theorem we have;

$$|z| = \sqrt{x^2 + y^2}$$

Example: Given $z = 2 + 5i$ determine $|z|$

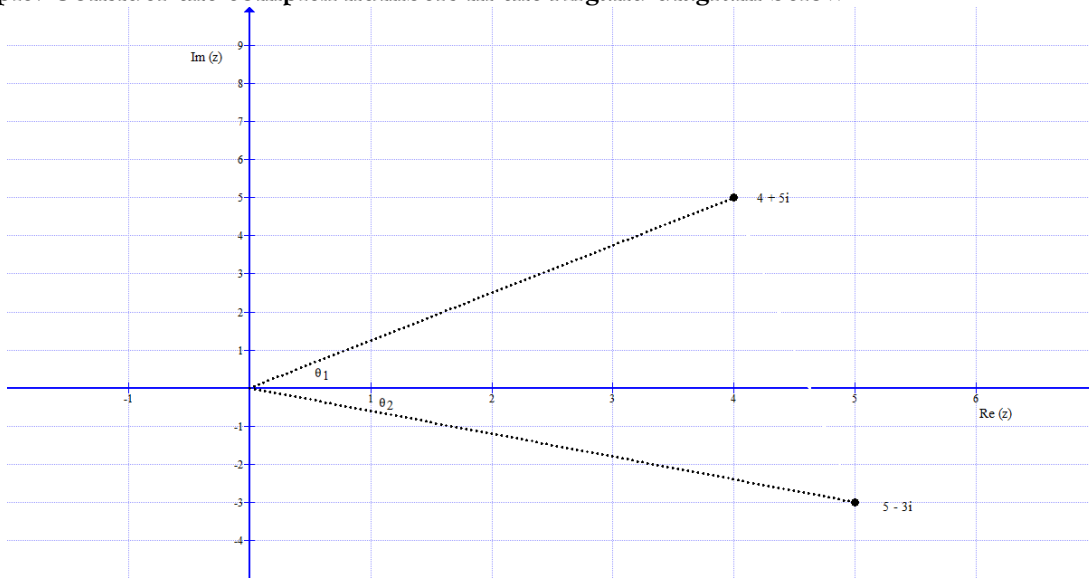
Solution: $|z| = \sqrt{2^2 + 5^2} = \sqrt{4 + 25} = \sqrt{29}$

Definition 10: (Argument)

The argument denoted θ or $\arg z$, $\arg (x + iy)$ or amplitude z is the magnitude of the angle between real axis and line representing the complex number in the Argand diagram.

Remark: For $x \neq 0$, $\tan\theta = \frac{y}{x}$

Example: Consider the complex numbers in the Argand diagram below

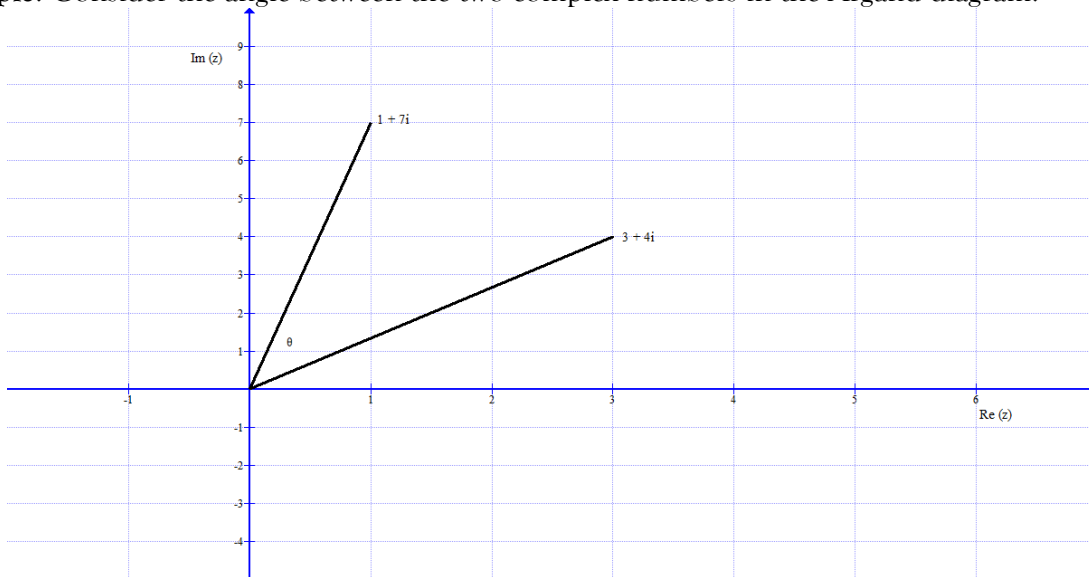


Given $z = 4 + 5i$ then $\tan \theta = \frac{5}{4} = 1.25 \Rightarrow \theta = \tan^{-1}(1.25) \approx 51.34^\circ$

Given $z = 5 - 3i$ then $\tan \theta = \frac{3}{5} \Rightarrow \theta = \tan^{-1}\left(\frac{3}{5}\right) \approx 30.96^\circ$

Note that the complex number is in the fourth quadrant hence $\arg(z) \approx -30.96^\circ$

Example: Consider the angle between the two complex numbers in the Argand diagram.



Determine the angle between $(3 + 4i)$ and $(1 + 7i)$

For $3 + 4i$ we have $\tan \theta = \frac{4}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 53.13^\circ$

For $1 + 7i$ we have $\tan \theta = 7 \Rightarrow \theta = \tan^{-1}(7) \approx 81.87^\circ$

Hence the angle between the two is $\approx 81.87^\circ - 53.13^\circ = 28.74^\circ$

Definition 11: (Mod-arg form or polar coordinate form of complex numbers)

This will be covered in the lecture 2 on polar coordinates.

Exercise

- 1) Plot the following complex numbers on the Argand diagram:
 - i) $3i$
 - ii) $2 + 5i$
 - iii) $1 - 3i$
 - iv) $-2i$
 - v) $-5i + 7$
 - vi) $2 - 9i$
- 2) Determine the modulus of the following complex numbers;
 - i) $9i$
 - ii) $2 - 5i$
 - iii) $1 - 7i$
 - iv) $-2i$
 - v) $-5i + 16$
 - vi) $11 - 2i$
- 3) Simplify the following into the form $x + yi$;
 - i) $\frac{2}{3-2i}$
 - ii) $\frac{5-7i}{6-11i}$
 - iii) $\frac{9i}{2-23i}$
 - iv) $(3 - 7i)(3 + 7i)$
 - v) $\frac{11i-3}{3-17i}$
 - vi) $\frac{2+7i}{2+7i}$
 - vii) $\frac{7}{i} - 3$
 - viii) i^5
 - ix) $\frac{3i+2}{3-5i}$
- 4) Simplify the following expressions;
 - i) $(2i + 5) + (3 - 11i)$
 - ii) $(4 + 5i)(7 - 2i)$
 - iii) $(5 - 2i) - (3i + 7)$
 - iv) $(23 - 5i)(-9 - 7i)$
 - v) $(i + 3)(2 - 5i)i$
 - vi) $2i(3 - 17i)(5 + 3i)$
- 5) Determine the value of x and y given the following;
 - i) $\frac{x}{2+i} + \frac{y}{3-2i} = 7$
 - ii) $\frac{x}{3+2i} + \frac{y}{5-3i} = 7 + 3i$
 - iii) $(3 + i)^2 - 5(x - yi) = x + yi$
 - iv) $\frac{x}{-2-i} + \frac{iy}{i+7} = \frac{5}{3+i}$
- 6) Evaluate the following;
 - i) $\sqrt{2 + 7i}$
 - ii) $\sqrt{3 - 5i}$
 - iii) $\sqrt{7 + 12i}$
 - iv) $\sqrt{1 - 13i}$
- 7) Derive the formula for finding the square root of a complex number $x + iy$
- 8) Given $z_1 = 2 + 13i$ and $z_2 = 7 - 9i$ evaluate the following;
 - i) $z_1 z_2$
 - ii) $\overline{z_1 z_2}$
 - iii) $z_1 \overline{z_2}$
 - iv) $\overline{\overline{z_1} z_2}$
 - v) $\frac{z_1}{z_2}$
 - vi) $\frac{\overline{z_1}}{z_2 z_1}$

Bibliography

Kahenya, P. (2017). *Foundation Maths*. LAP Lambert Academic Publishers.

Stewart, J. (2012). *Calculus (7th ed.)*. BROOKS/COLE Cengage Learning.