Mathematics for Science

Lecture 5

Polynomials: Remainder and Factor Theorem

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Introduction to lecture 5

This lecture will introduce polynomial expressions and equations, Remainder and Factor theorems and their applications. Polynomial functions are important in calculus and other branches of mathematics e.g. abstract algebra that deal with polynomial codes among others

Intended learning outcomes

At the end of this lecture you will be able to;

(i) Define a polynomial expression, and/or equation.

(ii) State and prove the remainder and factor theorems.

(iii) Apply remainder and factor theorems.

References

These lecture notes should be complemented with relevant topics in (Kahenya, 2017; Murray & Robert, 2009; Stewart, 2012).

Definition 1: Polynomials are algebraic expressions of the form;

$$a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_nx^n$$

with known real or complex numbers $a_0, a_1, \cdots, a_n$ and in variable x.

The real or complex numbers $a_0, a_1, a_2, \cdots, a_n$ are referred to as the coefficients of the respective variable. Note that $x^0 = 1$ hence $a_0x^0 = a_0$ i.e. it is a constant.

All the powers or exponent of the variable x must be whole numbers i.e., $0, 1,2,3, \cdots, n \in N$ (or 0 and positive integers).

Example 1: The following algebraic expressions are polynomials;

$$2x^2 + 3x + 4; x^3 + 2x^2 + 5x - 9$$

Example 2: The following are not polynomials;

$$\frac{1}{x^2} + \frac{1}{x} + 1; x^{-5} + x^{-2} + 3; \frac{1}{1 + y + 3y^2}$$

Definition 2: (Type of polynomials)

Polynomials of degree 1 are referred to as linear i.e. $a_0 + a_1x$

Quadratics are polynomials of degree 2 i.e. expressions of the form $a_2x^2 + a_1x + a_0$ are polynomials of degree 2.

Polynomials of degree 3 are cubic polynomials i.e. $a_0 + a_1x + a_2x^2 + a_3x^3$

In general, a polynomial of degree n is of the form; $a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_nx^n$

A polynomial of degree n in the variable x is a function of the form;

$$p(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_nx^n, \text{with } a_n \neq 0 \text{ and } n \in \mathbb{Z}^+$$

Any value of x that makes $p(x)$ disappear is referred to as a root of the equation $p(x) = 0$

Definition 3: (Rational integral equation)

The equation $p(x) = 0$ is called a rational integral equation of degree n in x.

Theorem 1: Fundamental theorem of algebra

Every polynomial equation $p(x) = 0$ has at least one root, a Real $\mathbb{R}$ or a Complex number $\mathbb{C}$.

Proposition 1: Every rational integral equation $p(x) = 0$ of n degree has exactly n roots.

Example 1: A quadratic equation $ax^2 + bx + c = 0$ has two roots.

A cubic equation $a + bx + cx^2 + dx^3 = 0$ has 3 roots.

A quartic equation $a + bx + cx^2 + dx^3 + ex^4 = 0$ has 4 roots and so on.

Definition 4: (The Division Algorithm) Let $a$be an integer and d a positive integer. Then there are unique integers q and r with $0 \leq r < d$ such that $a = dq + r$ where d is the divisor, q is the quotient and r is the remainder.

This can be applied to polynomials such that if a polynomial $P(x)$ is divided by a linear polynomial $(x - \alpha)$ where α is a constant we can get;

$$P(x) = Q(x)(x - \alpha) + R(x)$$

with $Q(x)$ is the quotient polynomial and $R(x)$ is the remainder

Theorem 2: (Remainder theorem) Given a polynomial function in x i.e. $p(x)$ of degree n. If it is divided by a linear function (polynomial), $x - \alpha$ such that the remainder R is a constant, then it (remainder) will be equal to $p(\alpha)$.That is; $R = p(\alpha)$

Proof: Applying the division algorithm we have; $p(x) = (x - \alpha)Q(x) + R \cdots (i)$

Suppose $x = \alpha$ then the above equation (i) becomes; $p(\alpha) = (\alpha - \alpha)Q(\alpha) + R$

$$p(\alpha) = 0 \cdot Q(\alpha) + R$$

$$p(\alpha) = R$$

Proposition 2: If $p(\alpha) = 0$ i.e. the Remainder $R = 0$ then α is called the root of $p(x)$.

Theorem 3: (Factor theorem). It is a result of Remainder theorem above. If a polynomial $p(x)$ is divided by a polynomial $(x - \alpha)$ and $p(\alpha) = 0$ implying that the remainder $R = 0$. Therefore the polynomial $(x - \alpha)$ must be a factor of $p(x)$.

Conversely, if $(x - \alpha)$ is a factor of $p(x)$ then $p(\alpha) = 0$ (i.e. α is a root of $p(x)$).

Exercise: Prove the Factor theorem.

Example 1: Show that $(x - 2)$ is a factor of $x^4 - x^2 - 11x + 10 = 0$

Solution: by Factor theorem if $(x - 2)$ is a factor then 2 is a root and $p(2) = 0 = R$ (remainder should be 0).

Therefore we have; $p(2) = 2^4 - 2^2 - 11(2) + 10 = 16 - 4 - 22 + 10 = 0$

Indeed $(x - 2)$ is a factor.

Example 2: Determine if $(x + 2)$ is a factor of $x^3 + x^2 - 2x + 1 = 0$

Solution: If $(x + 2)$ is a factor then -2 is root and $R = 0$, otherwise it is not.

Therefore $p(-2) = (-2)^3 + (-2)^2 - 2(-2) + 1 = -8 + 4 + 4 + 1 = 1$

The remainder is 1 and hence -2 is not a root. Therefore $(x + 2)$ is not a factor of $x^3 + x^2 - 2x + 1 = 0$.

Example 3: Use the Remainder theorem to find the remainder when $3x^3 - 4x^2 + x + 5$ is divided by $(x - 3)$.

Solution: by definition, $R = p(\alpha)$ when $p(x)$ is divided by $(x - \alpha)$

In our case $\alpha = 3$ then $p(3) = 3(3)^3 - 4(3)^2 + 3 + 5 = 81 - 36 + 3 + 5 = 53$.

The remainder $R = 53$.

Example 4: Find the numerical value of k if the expression $x^3 + x^2 + 4k - 1$ has a remainder 3 when divided by $(x - 2)$.

Solution: by definition, $R = p(\alpha)$ when $p(x)$ is divided by $(x - \alpha)$

In our case; $\alpha = 2, R = 3$

Therefore $p(\alpha) = p(2) = 8 + 4 + 4k - 1 = 3$

$$\Rightarrow 11 + 4k = 3 \Rightarrow 4k = -8 \therefore k = -2$$

Example 5: Use the remainder theorem to find the remainder R when $p(x) = 4x^3 - x^2 + 4x - 8$ is divided by $(4x - 1)$.

Solution: Note that when $p(x)$ is divided by $(\beta x - \alpha)$ then the remainder $R = p\left(\frac{\alpha}{\beta}\right)$.

In our case we have $(\beta x - \alpha) = (4x - 1)$

$$\Rightarrow R = p\left(\frac{1}{4}\right) = 4\left(\frac{1}{4}\right)^3 - \left(\frac{1}{4}\right)^2 + 4\left(\frac{1}{4}\right) - 8$$

$$= \frac{1}{16} - \frac{1}{16} + 1 - 8 = -7$$

Definition 5: (Long division) One can use the remainder theorem to find the remainder. Alternatively one can use the long division to find the remainder when a polynomial $p(x)$ is divided by another polynomial which is not necessarily linear.

Example 6: Determine the remainder when $p(x) = 2x^3 + 2x + 2$ is divided by $(x + 1)$ using the long division.

Solution:

$$\begin{array}{r}
2x^2 - 2x + 4 \\
x+1\overline{)\ 2x^3 \qquad\quad + 2x + 2} \\
\underline{-\,2x^3 - 2x^2} \qquad\qquad \\
-\,2x^2 + 2x \qquad \\
\underline{2x^2 + 2x} \qquad \\
4x + 2 \\
\underline{-\,4x - 4} \\
-\,2
\end{array}$$

The remainder is -2

Example 7: Use the long division to find the remainder when $x^3 - x - 2$ is divided by $(x - 2)$.

$$\begin{array}{r l}
 & \quad\quad\quad\quad x^2 + 2x + 3 \\ \cline{2-2}
x - 2) & x^3 \quad\quad\quad\quad - x + 2 \\
 & -x^3 + 2x^2 \\ \cline{2-2}
 & \quad\quad 2x^2 \; - x \\
 & \quad - 2x^2 + 4x \\ \cline{2-2}
 & \quad\quad\quad\quad\quad 3x + 2 \\
 & \quad\quad\quad\quad - 3x + 6 \\ \cline{2-2}
 & \quad\quad\quad\quad\quad\quad\quad 8
\end{array}$$

The remainder is 8

Definition 6: (Synthetic Division)

Determine the remainder and the quotient when $p(x) = 2x^3 + 2x + 2$ is divided by $(x + 1)$ using the synthetic division.

Step 1: Write down the dividend in descending powers of the variable and where the term is missing we use zero for the coefficients i.e. $2x^3 + 0x^2 + 2x + 2$

Step 2: Write the divisor in the form $x - a$ i.e. $x + 1 = x - (-1)$

Step 3: Write -1 to the left and the coefficients of the dividend to the right

$$\begin{array}{r|rrrr}
-1 & 2 & 0 & 2 & 2 \\ \hline
\end{array}$$

Step 4: Drop down the first term of the dividend to the 3rd row (such that you leave a blank row above it i.e.

$$\begin{array}{r|rrrr}
 & 2 & 0 & 2 & 2 \\
-1 & & & & \\ \hline
 & 2 & & &
\end{array}$$

Step 5: Multiply -1 with the 2 to -2 and write the product below 0. Then add 0 to -2 and write the sum in the third row. Repeat the process to get;

$$\begin{array}{r|rrrr}
 & 2 & 0 & 2 & 2 \\
-1 & & -2 & 2 & -4 \\ \hline
 & 2 & -2 & 4 & -2
\end{array}$$

From the last row we can see that the remainder is -2 and the quotient is $2x^2 - 2x + 4$

Example 8: Determine the remainder when $p(x) = 2x^3 + x^2 + 2x - 2$ is divided by $(x + 2)$ using the synthetic division.

Solution: Note that the coefficients of the dividend are 2, 1, 2, -2 respectively, and $a = -2$

Therefore we have;

	2	1	2	−2
−2		−4	6	−16
	2	−3	8	−18

From the 3rd row the remainder $R = -18$ and the quotient is $2x^2 - 3x + 8$

Proposition 3: If a complex number z is a root of a rational integral equation $p(x) = 0$ that has real coefficients, then the complex conjugate $\bar{z}$ is also a root.

Example 1: Determine the other roots of $x^3 - x^2 + x + 39 = 0$ if -3 is a root.

Solution: We can show that indeed -3 is a root i.e.

$p(-3) = 0 => (-3)^3 - (-3)^2 - 3 + 39 = -27 - 9 - 3 + 39 = 0$

We then use synthetic division to find the quotient when $p(x)$ is divided by $(x + 3)$ to get;

	1	−1	1	39
−3		−3	12	−39
	1	−4	13	0

The quotient is $x^2 - 4x + 13$. To get the roots we use the quadratic formula to get;

$$x = \frac{4 \pm \sqrt{16 - 4 \cdot 1 \cdot 13}}{2} = \frac{4 \pm i\sqrt{36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

The roots are $(2 + 3i)$ and $(2 - 3i)$ which are complex conjugates of each other.

Theorem 4: (Integral root theorem)

If $p(x) = 0$ has integral coefficients with the lead coefficient as 1 i.e.

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0 = 0$$

Then any rational root is an integer and a factor of a_0

Example 1: Determine the factors of $x^3 - 9x^2 + 21x - 5$

Solution: The rational roots (if any) are limited to the factors of 5 i.e. $\pm 1, \pm 5$

We can show that 5 is a root i.e. $p(5) = 0$

$$p(5) = 5^3 - 9(25) + 21(5) - 5 = 0$$

Hence $(x-5)$ is a factor. We can then use synthetic division to find the quotient and then the other factors.

$$\begin{array}{r|rrrr} & 1 & -9 & 21 & -5 \\ 5 & & 5 & -20 & 5 \\ \hline & 1 & -4 & 1 & 0 \end{array}$$

The quotient is x^2-4x+1 we apply quadratic formula to get the other roots;

$$x=\frac{4\pm\sqrt{16-4}}{2}=\frac{4\pm2\sqrt{3}}{2}=2\pm\sqrt{3}$$

The other factors are $(x-2-\sqrt{3})$ and $(x-2+\sqrt{3})$

Proposition 4: If the rational integral equation $\mathrm{p}(\mathrm{x})=0$ with rational coefficients has; $\mathrm{p}+\sqrt{\mathrm{q}}$ as root then $\mathrm{p}-\sqrt{\mathrm{q}}$ is also a root.

Example 1: In the previous example we saw that $(2+\sqrt{3})$ and $(2-\sqrt{3})$ are both roots of; $\mathrm{x}^3-9\mathrm{x}^2+21\mathrm{x}-5=0$

Theorem 5: (Rational root theorem)

If $\frac{b}{c}$ with $(\mathrm{b},\mathrm{c})=1$ is a root of the equation $\mathrm{a_n x^n}+\cdots+\mathrm{a_2x^2}+\mathrm{a_1x}+\mathrm{a_0}=0, \mathrm{a_n}\neq0$ with integral coefficients, then b is a factor of $\mathrm{a_0}$ and c is a factor of $\mathrm{a_n}$.

Example 1: Suppose $\frac{\mathrm{b}}{\mathrm{c}}$ is a rational root of $8\mathrm{x}^3+3\mathrm{x}^2-5\mathrm{x}+9=0$ then the values of b are limited to the factors of 9 i.e. $\pm1,\pm3,\pm9$ and the values of c are limited to the factors of 8 i.e. $\pm1,\pm2,\pm4,\pm8$.

Therefore the possible rational roots are limited to;

$\pm1,\pm\frac{1}{2},\pm\frac{1}{4},\pm\frac{1}{8}\pm2,\pm3,\pm\frac{3}{2},\pm\frac{3}{4},\pm\frac{3}{8},\pm9,\pm\frac{9}{2},\pm\frac{9}{4},\pm\frac{9}{8}$

Example 2: Find the roots of $2x^3 - x^2 - 13x - 6 = 0$

Solution: Note that the rational root $\frac{b}{c}$ is limited to $\pm 1, \pm\frac{1}{2}, \pm 2, \pm 3, \pm\frac{3}{2}, \pm 6$

Again if we divide every term by 2 to get $x^3 - \frac{1}{2}x^2 - \frac{13}{2}x - 3 = 0$ then the integral root can be ± 1 or ± 3.

We can quickly test to see if 1 is a root i.e. $p(1) = 2 - 1 - 13 - 6 \neq 0$ hence 1 is not a root.

We can test with 3 to see if 3 is indeed a root i.e. $p(3) = 54 - 9 - 39 - 6 = 0$. 3 is a root and therefore $(x - 3)$ is a factor.

Next we use the synthetic division to get the quotient to enable us to find the other factors.

	2	−1	−13	−6
3		6	15	6
	2	5	2	0

The quotient is $2x^2 + 5x + 2$.

Applying quadratic equation to get the roots i.e. $x = \frac{-5 \pm \sqrt{25 - 4\cdot 2\cdot 2}}{4} = \frac{-5 \pm 3}{4} = -2 \text{ or } -\frac{1}{2}$

Therefore the roots of $2x^3 - x^2 - 13x - 6 = 0$ are $\{3, -2, -\frac{1}{2}\}$.

Determining a polynomial given its roots (Degree 3)

We shall restrict our working to polynomial of degree 3.

Definition 1: Let $\alpha, \beta, \text{and } \mu$ be the roots of the polynomial $ax^3 + bx^2 + cx + d = 0$ then $x = \alpha \text{ or } x = \beta \text{ or } x = \mu$.

Consequently the factors of the polynomial are $(x - \alpha), (x - \beta) \text{and } (x - \mu)$

We can therefore we have; $(x - \alpha)(x - \beta)(x - \mu) = ax^3 + bx^2 + cx + d \cdots \text{(i)}$

Expanding the LHS of (i) and writing it as a standard polynomial in x we have;

$$x^3 - (\alpha + \beta + \mu)x^2 + (\alpha\beta + \alpha\mu + \beta\mu)x - \alpha\beta\mu \cdots \text{(ii)}$$

We write the RHS of (i) with the coefficient of $x^3 = 1$ i.e.

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} \cdots \text{(iii)}$$

Equation (i) becomes;

$$x^3 - (\alpha + \beta + \mu)x^2 + (\alpha\beta + \alpha\mu + \beta\mu)x - \alpha\beta\mu = x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}$$

Equating the LHS and RHS we get;

Sum of the roots; $\alpha + \beta + \mu = -\frac{b}{a} \cdots (*)$

Sum of the product of pair of roots; $\alpha\beta + \alpha\mu + \beta\mu = \frac{c}{a} \cdots (**)$

Product of roots; $\alpha\beta\mu = -\frac{d}{a} \cdots (***)$

The above (*) equations are used to get a polynomial given its roots.

Example 1: The roots of a polynomial are 2, -3, and 5. Determine the polynomial in x.

Solution: We know that given the roots $\alpha, \beta,$ and μ then

$$x^3 - (\alpha + \beta + \mu)x^2 + (\alpha\beta + \alpha\mu + \beta\mu)x - \alpha\beta\mu = 0$$

Therefore we get; $x^3 - (2 - 3 + 5)x^2 + (-6 + 10 - 15)x - (2)(-3)(5) = 0$

$$x^3 - 4x^2 - 21x + 30 = 0$$

Example 2: Suppose the roots of $5x^3 + x^2 - 3x + 5 = 0$ are $\alpha, \beta,$ and μ. Determine without solving $5x^3 + x^2 - 3x + 5 = 0$ a polynomial whose roots are $\alpha - 2, \beta - 2, \mu - 2$.

Solution: Sum of the roots $\alpha + \beta + \mu = -\frac{b}{a} = -\frac{1}{5}$; Sum of the products of the roots $\alpha\beta + \alpha\mu + \beta\mu = \frac{c}{a} = -\frac{3}{5}$; Product of the roots $\alpha\beta\mu = -\frac{d}{a} = -1$

Therefore for the polynomial with roots $\alpha - 2, \beta - 2, \mu - 2$ we have;

$$(\alpha - 2) + (\beta - 2) + (\mu - 2) = (\alpha + \beta + \mu) - 6 = -\frac{1}{5} - 6 = -\frac{31}{5}$$

Next we find the sum of the products of the roots;

$$(\alpha - 2)(\beta - 2) + (\alpha - 2)(\mu - 2) + (\beta - 2)(\mu - 2) = \beta\mu + \alpha\mu + \alpha\beta - 4\beta - 4\alpha - 4\mu + 12$$

$$= -\frac{3}{5} + 12 - 4(\alpha + \beta + \mu) = \frac{57}{5} + \frac{4}{5} = \frac{61}{5}$$

The product of the roots is; $(\alpha - 2)(\beta - 2)(\mu - 2) = \alpha\beta\mu - 2\alpha\beta - 2\alpha\mu - 2\beta\mu + 4\alpha + 4\beta + 4\mu - 8 = -1 - 2\left(-\frac{3}{5}\right) + 4\left(-\frac{1}{5}\right) - 8 = -1 + \frac{6}{5} - \frac{4}{5} - 8 = -\frac{43}{5}$

Therefore our polynomial is; $x^3 - (\alpha + \beta + \mu)x^2 + (\alpha\beta + \alpha\mu + \beta\mu)x - \alpha\beta\mu = 0$

$$= x^3 - \left(-\frac{31}{5}\right)x^2 + \frac{61}{5}x - \left(-\frac{43}{5}\right) == x^3 + \frac{31}{5}x^2 + \frac{61}{5}x + \frac{43}{5} = 0$$

$$= 5x^3 + 31x^2 + 61x + 43 = 0$$

Exercise

1) Find the reminder when the expressions below are divided by $(x - 2)$ and $(3x + 2)$.

a) $x^3 + 2x + 1$

b) $4x^4 + 2x^3 - x^2 - 3x - 2$

c) $x^5 - x^7 - 9$

d) $x^4 - x^3 + x^2 + x - 1$

2) Use the remainder theorem to determine the remainder when the following expressions are divided by $(x - 1), (2x + 1), (x + 3)$.

a) $x^{23} + x^{20} - x + 1$

b) $x^3 + x^7 - 2x^2 + 1 - 2x$

c) $3x^2 + 2x + 5$

d) $2x^3 - 2x^2 - 7x + 12$

3) Find the numerical value of a if the $x^3 + 2x^2 - 3ax + 2$ is divisible $(x + 1)$.

4) Show that $(x + 3)$ is not a factor of $x^6 - 9$

5) Determine the roots of the expression $x^3 - 2x^2 + 3x + n$ if $(x - 2)$ is a factor.

6) Suppose $(x + 2)$ and $(x - 5)$ are factors of $x^4 - ax^3 + 5x^2 + bx - 30$. Find the numerical value of a and b.

7) Apply synthetic division to determine the remainder and the quotient of the following expressions when divided by; $(x + 2)$, $(2x - 3)$, and $(3x - 5)$.

a) $x^3 + 2x^2 - x + 3$

b) $5x^4 + x^2 - 3x - 9$

c) $x^7 + x^3 - 12$

d) $x^4 + 2x^3 + x^2 - 2x + 8$

8) Suppose the roots of $3x^3 + 2x^2 + 2x - 6 = 0$ are $\alpha, \beta,$ and μ. Determine without solving of $3x^3 + 2x^2 + 2x - 6 = 0$ a polynomial whose roots are $\alpha - 1, \beta - 1, \mu - 1$.

9) Suppose the roots of $x^3 - 5x^2 - 3x - 7 = 0$ are $\alpha, \beta,$ and μ. Determine without solving of $x^3 - 5x^2 - 3x - 7 = 0$ a polynomial whose roots are $\alpha + 2, \beta + 2, \mu + 2$.

Bibliography

Kahenya, P. (2017). *Foundation Maths*. LAP Lambert Academic Publishers.

Murray, S., & Robert, M. (2009). *College Algebra*. McGraw-Hill.

Stewart, J. (2012). *Calculus* (7th ed.). BROOKS/COLE Cengage Learning.