Mathematics for Science

Lecture 6

Decomposing Rational Functions: CASE I and II

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Introduction to lecture 6

This lecture will introduce Rational functions and their decomposition. In Lecture 5 we discussed polynomials and the application of the remainder and factor theorems. Decomposing rational fractions is applicable when integrating rational functions in calculus.

Intended learning outcomes

At the end of this lecture you will be able to;

(i) Define a rational function.

(ii) Decompose rational functions cases I and II.

Further reading

These lecture notes should be complemented with relevant topics in (Kahenya, 2017; Murray & Robert, 2009; Stewart, 2012).

Murray & Robert (2009) have more additional details on rational functions especially on the asymptotes and graphing of rational functions while Stewart (2012) has application of decomposition of rational functions in integration.

Definition 1: (Rational functions)

A rational function is a ratio of two polynomials. If we have two polynomials $p(x)$ and $q(x)$ then a rational function $R(x)$ is of the form;

$$R(x) = \frac{p(x)}{q(x)} \text{ with } q(x) \neq 0, \big(p(x), q(x)\big) = 1$$

Examples of rational functions are; $\frac{2x^3+3x^2+1}{x^5+2x^2+3}; \frac{3x}{4x^2+3}; \frac{1}{x+3}$

Definition 2: (Proper and improper fractions)

A proper fraction is where the degree of the numerator $p(x)$ is less than the degree of the denominator $q(x)$, other it is an improper fraction.

Examples of proper fractions are; $\frac{2x}{3x^2+2x-1}$; $\frac{3-x}{x^3-16}$

Examples of improper fractions are; $\frac{3x^2+2x}{x+2}$; $\frac{x^3-x^2-2x+1}{5x^2+2x+1}$

Definition 3: (Partial fractions)

A proper fraction may be written as a sum of other proper fractions called partial fractions, whose denominators are of lower degree than the denominator of the given proper fraction.

An improper fraction may be written as a sum of a polynomial (a whole) and a proper fraction.

Definition 4: (Rules of decomposing a proper fraction)

The factors of the denominator determine the decomposition of proper fractions.

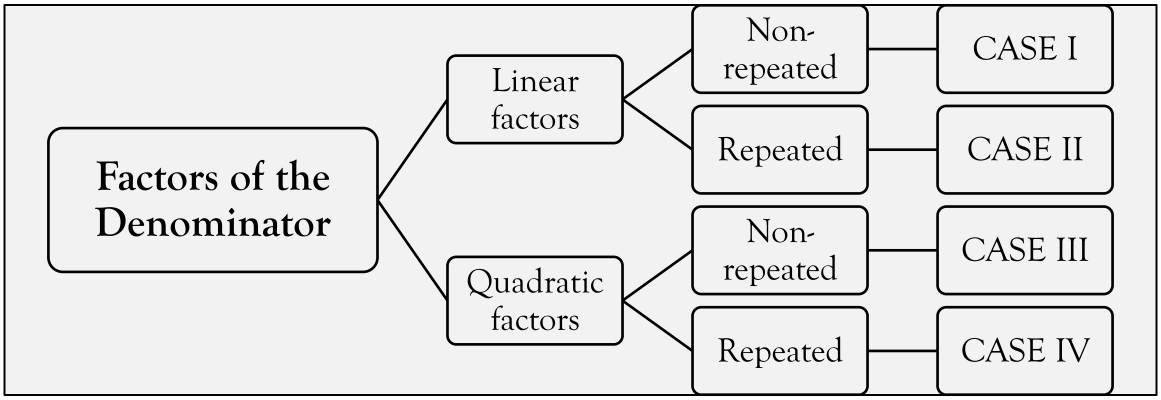
Rule 1: If the denominator can be factorized into linear factors that occurs once, then we have what we call CASE I.

Rule 2: If the denominator can factorize into linear factors with some being repeated then we call this CASE II

Rule 3: If the denominator can be factorized into quadratic factors that occurs once, then we have what we call CASE III.

Rule 4: If the denominator can factorize into quadratic factors with some being repeated then we call this CASE IV.

This lecture will explore CASES I and II.



CASE I: The denominator $q(x)$ is a product of unique (non-repeated) linear factors

This is the case where the denominator can be factorized into distinct or unique linear factors i.e.

$$q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_nx + b_n)$$

Then given the rational fraction $\frac{p(x)}{q(x)}$ there exists constants $A_1, A_2, \cdots A_n$ such that

$$\frac{p(x)}{q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

Example 1: Decompose into partial fractions; $\frac{3x+5}{(x+2)(x-1)}$

Solution: The denominator is a product of two distinct linear factors $(x - 1)$ and $(x + 2)$

Hence the rational fraction can be written as; $\frac{3x+5}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$

The RHS can be simplified to; $\frac{3x+5}{(x+2)(x-1)} = \frac{A(x-1)+B(x+2)}{(x+2)(x-1)} \Rightarrow 3x + 5 \equiv A(x - 1) + B(x + 2) \cdots (i)$

If we let $x = 1$, A will explode, and we can get B i.e. $3(1) + 5 \equiv B(1 + 2) = 3B$

$$8 = 3B \therefore B = \frac{8}{3}$$

If we let $x = -2$, B will explode, and we can get A i.e. $3(-2) + 5 = A(-2 - 1) = -3A$

$$-1 = -3A \therefore A = \frac{1}{3}$$

Hence our fraction can be decomposed into the sum of the partial fractions as below;

$$\frac{3x + 5}{(x + 2)(x - 1)} = \frac{1}{3(x + 2)} + \frac{8}{3(x - 1)}$$

Example 2: Decompose into partial fractions;

$$\frac{x^2 + 2x + 1}{2x^3 - x^2 - 3x}$$

Solution: We need to factorize the denominator i.e.

$$2x^3 - x^2 - 3x = x(2x^2 - x - 3)$$

$$2x^2 - x - 3 = 2x^2 + 2x - 3x - 3 = 2x(x+1) - 3(x+1) = (2x-3)(x+1)$$

Hence our fraction becomes;

$$\frac{x^2 + 2x + 1}{x(x+1)(2x-3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{2x-3} = \frac{A(x+1)(2x-3) + Bx(2x-3) + Cx(x+1)}{x(x+1)(2x-3)}$$

$$=> x^2 + 2x + 1 \equiv A(x+1)(2x-3) + Bx(2x-3) + Cx(x+1) \cdots (i)$$

If $x = 0$, B and C in equation (i) will explode to get;

$$1 = A(1)(-3) = -3A \therefore A = -\frac{1}{3}$$

If $x = -1$, A and C in equation (i) will explode to get;

$$1 - 2 + 1 = B(-1)[2(-1) - 3]$$

$$0 = 5B \therefore B = 0$$

If $x = \frac{3}{2}$, A and B in equation (i) will explode to get;

$$\frac{9}{4} + 3 + 1 = C\left(\frac{3}{2}\right)\left[\frac{3}{2} + 1\right] = 3\frac{3}{4}C$$

$$\Rightarrow \frac{25}{4} = \frac{15}{4}C \therefore C = \frac{5}{3}$$

Hence our fraction becomes;

$$\frac{x^2 + 2x + 1}{x(x+1)(2x-3)} = \frac{5}{3(2x-3)} - \frac{1}{3x}$$

Example 3: Decompose the following fraction;

$$\frac{3x^3 + 2x^2 - 3x + 6}{x(x+2)}$$

Solution: Note the fraction is an improper function. The degree of $p(x) >$ degree of $q(x)$.

We can use long division or synthetic division to convert it to a mixed fraction

$$
\begin{array}{r}
3x - 4 \\
x^2 + 2x \overline{\big)\; 3x^3 + 2x^2 - 3x + 6} \\
-\,3x^3 - 6x^2 \qquad\qquad \\
\hline
-\,4x^2 - 3x \quad \\
4x^2 + 8x \quad \\
\hline
5x + 6
\end{array}
$$

$$\frac{3x^3 + 2x^2 - 3x + 6}{x(x+2)} = (3x - 4) + \frac{5x + 6}{x^2 + 2x}$$

Next we decompose the proper fraction $\frac{5x+6}{x(x+2)}$ to get

$$\frac{5x + 6}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2} = \frac{A(x+2) + Bx}{x(x+2)}$$

$$\Rightarrow 5x + 6 = A(x+2) + Bx$$

Let $x = 0$ then $6 = 2A \therefore A = 3$

Let $x = -2$ then $-4 = -2B \therefore B = 2$

Therefore our fraction can be decomposed into;

$$\frac{3x^3 + 2x^2 - 3x + 6}{x(x+2)} = (3x - 4) + \frac{3}{x} + \frac{2}{x+2}$$

CASE II: The denominator $\mathbf{q(x)}$ is a product of linear factors, with some being repeated

This is the case where the denominator can be factorized into distinct or unique linear factors with some being repeated i.e.

$$q(x) = (a_1x + b_1)(a_2x + b_2)^r$$

where $(a_2x + b_2)$ is repeated r times.

Then given the rational fraction $\frac{p(x)}{q(x)}$ there exists constants $A_1, A_2, \cdots A_n$ such that

$$\frac{p(x)}{q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_2x + b_2)} + \frac{A_3}{(a_2x + b_2)^2} + \cdots + \frac{A_k}{(a_2x + b_2)^r}$$

Example 1: Decompose into partial fractions;

$$\frac{x^2+5}{(x-3)(x+1)^2}$$

Solution: The denominator consists of two linear factors; a unique factor $(x-3)$ and $(x+1)$ repeated. Hence we have

$$\frac{x^2+5}{(x-3)(x+1)^2}=\frac{A}{x-3}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$\frac{(x^2+5)}{(x-3)(x+1)^2}=\frac{A(x+1)^2+B(x-3)(x+1)+C(x-3)}{(x-3)(x+1)^2}$$

$$\Rightarrow x^2+5\equiv A(x+1)^2+B(x-3)(x+1)+C(x-3)$$

$$x^2+5\equiv(B+A)x^2+(C-2B+2A)x+(A-3C-3B)$$

Equating both sides we get;

$$B+A=1\cdots(i)$$

$$C-2B+2A=0\cdots(ii)$$

$$A-3C-3B=5\cdots(iii)$$

From (i) $A=1-B$.

Replacing A in equation (ii) to get; $C-2B+2(1-B)=0\Rightarrow\ C-4B=-2\therefore C=4B-2$

Replacing A and C in equation (iii) to get; $(1-B)-3(4B-2)-3B=5$

$$1-B-12B+6-3B=5$$

$$-16B=-2\therefore B=\frac{1}{8}$$

Therefore $A=1-B=1-\frac{1}{8}=\frac{7}{8}$; $C=4B-2=\frac{1}{2}-2=-\frac{3}{2}$

The decomposition of the fraction is;

$$\frac{x^2+5}{(x-3)(x+1)^2}=\frac{7}{8(x-3)}+\frac{1}{8(x+1)}-\frac{3}{2(x+1)^2}$$

Example 2: Decompose into partial fractions;

$$\frac{2x^2 - 5x + 6}{x(x+3)(x-2)^3}$$

Solution: Note that the denominator consists of three linear factors; x, $(x+3)$ and $(x-2)$. However $(x-2)$ is repeated.

$$\frac{2x^2 - 5x + 6}{x(x+3)(x-2)^3} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-2} + \frac{D}{(x-2)^2} + \frac{E}{(x-2)^3}$$

The RHS can be simplified to;

$$= \frac{A(x+3)(x-2)^3 + Bx(x-2)^3 + Cx(x+3)(x-2)^2 + Dx(x+3)(x-2) + Ex(x+3)}{x(x+3)(x-2)^3}$$

Therefore we have;

$$2x^2 - 5x + 6 \equiv A(x+3)(x-2)^3 + Bx(x-2)^3 + Cx(x+3)(x-2)^2 + Dx(x+3)(x-2) + Ex(x+3)$$

Let $x = 0$ then B, C,D, and E will explode to get;

$$6 = A(3)(-2)^3 => 6 = -24A \therefore A = -\frac{1}{4}$$

Let $x = 2$ then A, B, C, and D will explode to get;

$$4 = E(2)(5) = 10E \therefore E = \frac{2}{5}$$

Let $x = -3$ then A, C, D, and E will explode to get;

$$39 = B(-3)(-3-2)^3 = 375B \therefore B = \frac{39}{375}$$

This method is weak. Since it will not be possible to get the values of C and D.

We need to expand the RHS and write it in the general format of a polynomial i.e.

$$= (C + B + A)x^4 + (D - C - 6B - 3A)x^3 + (E + D - 8C + 12B - 6A)x^2 + (3E - 6D + 12C - 8B + 28A)x - 24A$$

Equating the coefficients of the x on the RHS to those on the LHS we have;

$$C + B + A = 0 \cdots (i)$$

$$D - C - 6B - 3A = 0 \cdots (ii)$$

$$E + D - 8C + 12B - 6A = 2 \cdots (iii)$$

$$3E - 6D + 12C + 28A = -5 \cdots (iv)$$

$$-24A = 6 \therefore A = -\frac{1}{4} \cdots (v)$$

Next we need to solve the 4 equations simultaneously.

Since we already have the values of A, B, and E we can get the value of C by considering equation (i)

$$C + B + A = 0 \Rightarrow C = -B - A = \frac{1}{4} - \frac{39}{375} = \frac{73}{500}$$

From equation (ii) we have;

$$D - C - 6B - 3A = 0$$

$$D = C + 6B + 3A = \frac{73}{500} + \frac{234}{375} - \frac{3}{4} = \frac{1}{50}$$

Hence our fraction can be decomposed into partials fractions

$$\frac{2x^2 - 5x + 6}{x(x+3)(x-2)^3} = -\frac{1}{4x} + \frac{13}{125(x+3)} + \frac{73}{500(x-2)} + \frac{1}{50(x-2)^2} + \frac{2}{5(x-2)^3}$$

Example 2: Decompose into partial fractions

$$\frac{x^3 + 3x + 2}{(x+1)^2(x+3)^3}$$

Solution:

$$\frac{x^3 + 3x + 2}{(x+1)^2(x+3)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2} + \frac{E}{(x+3)^3}$$

The RHS can be simplified to

$$= \frac{A(x+1)(x+3)^3 + B(x+3)^3 + C(x+1)^2(x+3)^2 + D(x+1)^2(x+3) + E(x+1)^2}{(x+1)^2(x+3)^3}$$

Hence we have;

$$x^3 + 3x + 2 = A(x+1)(x+3)^3 + B(x+3)^3 + C(x+1)^2(x+3)^2 + D(x+1)^2(x+3) + E(x+1)^2$$

We can let $x = -1$ and this will eliminate A, C,D, and E to get;

$$-2 = 8B => B = -\frac{1}{4}$$

Next, let $x = -3$ and this will eliminate A, B, C, and D to get;

$$-27 - 9 + 2 = 4E \Rightarrow 4E = -34 \therefore E = -\frac{17}{2}$$

However it is no longer possible to get the other constants by the above elimination method.

Expanding and simplifying the RHS to get;

$$(C+A)x^4 + (D+8C+B+10A)x^3 + (E+5D+22C+9B+36A)x^2 + (2E+7D+24C+27B + 54A)x + E + 3D + 9C + 27B + 27A$$

We now equate the coefficients;

$$C + A = 0 \Rightarrow C = -A \cdots \text{(i)}$$

$$D + 8C + B + 10A = 1 \Rightarrow D + 8C + 10A$$

$$= D - 8A + 10A = \frac{5}{4}$$

$$= D + 2A = \frac{5}{4} \cdots \text{(ii)}$$

$$E + 5D + 22C + 9B + 36A = 0$$

$$\Rightarrow 5D + 22C + 36A = \frac{17}{2} - 9\left(-\frac{1}{4}\right) = \frac{17}{2} + \frac{9}{4} = \frac{43}{4}$$

$$= 5D - 22A + 36A$$

$$= 5D + 14A = \frac{43}{4}$$

$$\Rightarrow 20D + 56A = 43 \cdots \text{(iii)}$$

$$2E + 7D + 24C + 27B + 54A = 3$$

$$\Rightarrow 7D + 24C + 54A = 3 - 2\left(-\frac{17}{2}\right) - 27\left(-\frac{1}{4}\right) = 20 + \frac{27}{4} = \frac{107}{4}$$

$$= 7D - 24A + 54A$$

$$= 7D + 30A = \frac{107}{4}$$

$$\Rightarrow 28D + 120A = 107 \cdots \text{(iv)}$$

$$E + 3D + 9C + 27B + 27A = 2$$

$$\Rightarrow 3D + 9C + 27A = \frac{17}{2} - 27\left(-\frac{1}{4}\right) = \frac{17}{2} + \frac{27}{4}$$

$$= 3D - 9A + 27A = \frac{61}{4}$$

$$= 3D + 18A = \frac{61}{4}$$

$$\Rightarrow 12D + 72A = 61 \cdots \text{(v)}$$

From equation (iii) $D = \frac{43-56A}{20}$

Replacing D in equation (iv) to get

$$28\left(\frac{43-56A}{20}\right)+120A = 107$$

$$1204-1568A+2400A = 2140$$

$$832A = 936 \therefore A = \frac{936}{832} = \frac{9}{8}$$

$$\text{Hence } C = -\frac{9}{8}$$

But $D = \frac{43-56A}{20} = \frac{43-56\left(\frac{9}{8}\right)}{20} = \frac{43-63}{20} = -\frac{20}{20} = -1$

Our constants are;

$$A = \frac{9}{8}, B = -\frac{1}{4}, C = -\frac{9}{8}, D = -1, E = -\frac{17}{2}$$

Therefore the partial fractions are;

$$\frac{x^2+3x+2}{(x+1)^2(x+3)^3} = \frac{9}{8(x+1)} - \frac{1}{4(x+1)^2} - \frac{9}{8(x+3)} - \frac{1}{(x+3)^2} - \frac{17}{2(x+3)^3}$$

Exercise

1) Decompose into partial fractions;

a) $\frac{3x+2}{x(x+7)}$

b) $\frac{x-7}{(x+2)(x-6)}$

c) $\frac{3x^2+2x^2-5}{(x+2)(x-3)(2x+7)}$

d) $\frac{2x^2+3}{(3x+2)(x-5)}$

e) $\frac{7x^3+2x^2+8x-10}{x(x+9)}$

f) $\frac{1}{(x-2)(x-5)}$

2) Decompose into partial fractions

a) $\frac{x+2}{x(x+2)^3}$

b) $\frac{x^2+3x+2}{(x+3)(x-5)^3}$

c) $\frac{x^5+2x^3+2}{(3x-2)^3}$

d) $\frac{5x^2+7x-14}{x^2(x+2)^2(x-5)^3}$

e) $\frac{x+9}{(3x-7)^2}$

f) $\frac{x^2+3x+12}{(x+3)^2(x-9)^3}$

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