Course: Automata Theory

Lecture 3: Introduction to Logic and Truth Tables

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Lecture learning outcomes

At the end of the lecture you will be able to:

(i) Define a compound statement, a connective and a truth value of a statement.

(ii) Explain the various connectives used in statements.

(iii) Differentiate tautology propositions from contradictory propositions

(iv) Solve problems involving compound statements, using truth tables

3.0 Logic & Truth Tables

A computer may be programmed to make decisions based on whether certain statements **are true or false.**

For example, "**The number of students in this class exceeds 50"**. This statement can either be true or false. A statement is either true or false but not both.

The truth or falsity of a statement is called its **truth value**.

Truth Tables

A **truth table** lists whether something is true or false and as stated above, the act of negation changes all the true values into false and the false values to true as shown below.

P	¬P
True	False
False	True

To construct a truth table, a column for every proposition in the expression must be created, and for every possible true and false value, a row is created showing all the possible combination that a given set of propositions can take.

Example: For two propositions, a truth table for all the possible truth value combinations will have four rows for T/F possibilities and two columns for the propositions. In general, there will be **2^n rows for *n* different propositions**.

A	B
True	True
True	False
False	True
False	False

Logical operators

These are needed to link together propositions and build more complex logical expressions and they operate in the same way as +, -, ×, and ÷ operators used to link mathematical expressions. The basic logical operators, along with **negation**, are **conjunction**, **disjunction**, **conditional**, and **biconditional**.

Some statements are **compound statements**; meaning, they are composed of **sub-statements and various connectives**.

For example, "He is intelligent or studies very hard" is an implicit statement. A compound statement and sub-statements would be; "He is intelligent" **and** "He studies very hard"

The fundamental property of a compound statement is that its truth value is completely determined by the truth values of its sub-statements together with the way in which they are **connected to form the compound statement**.

Several connectives are used to connect sub-statements and these include conjunction, disjunction and negation.

(i.) Conjunction (P Λ Q)

Any two statements can be combined by the word "***and***" to form a compound statement called the *conjunction of the original statement* symbolically written as: P Λ Q; which denotes the conjunction of the statements P and Q read as "*P and Q*".

The truth values of the compound statement P Λ Q is given by the following truth table: -

P	Q	P Λ Q
False	False	False
False	True	False
True	False	False
True	True	True

Here, the last line is a short way of saying **that if P is true and If Q is True, then P Λ Q is also true.** The other lines have analogous meaning and this table defines precisely the truth values of the compound statement P Λ Q as a function of the truth values of P and Q.

Consider the following statements which are represented in form of a truth table below: -

i). Paris is in France and 2 + 2 = 4 *(True and True)*

ii). Paris is in France and 2 + 2 = 5 *(True and False)*

iii). Paris is in England and 2 + 2 = 4 *(False and True)*

iv). Paris is in England and 2 + 2 =5 *(False and False)*

P	Q	P Λ Q
True	True	True
True	False	False
False	True	False
False	False	False

(ii.) Disjunction (P V Q)

Any two statements can be combined by the word "***or***" to form a new statement which is called the *disjunction of the original two statements*, symbolically written as P V Q.

This denotes the disjunction of the statements P and Q which is read as "*P or Q*". The truth values of P V Q are given by the following truth table: -

P	Q	P V Q
True	True	True
True	False	True
False	True	True
False	False	False

P V Q is false only when both statements are false and is otherwise true.

(iii.) Negation (~P)

Given any statement P, another statement known as *the negation of P* can be formed by writing "***It is false that**…*" *before P* or if possible, by inserting in front of P the word "*not*"; symbolically written as "~P".

This denotes the negation of P read as "*not* P" and the truth values of ~P are given by the following table: -

P	~P
True	False
False	True

If P is true, then ~P is false, thus the truth value of the negation of any statement is always the opposite of the truth value of the original statement.

Propositions & Truth Tables

Through repetitive use of the logical connectives - Λ, V, ~ and others, we are able to construct a compound statement(s) that are more involving.

In the case where the sub-statements P, Q of compound statements P (P, Q …) are variables; we call the compound statement a **proposition**.

The truth value of a proposition depends exclusively upon the truth values of its variables i.e. the truth value of a proposition is known once the truth values of its variables are known.

A simple way to show its relationship is through a truth table and the truth table of the proposition ~(P Λ ~Q), for example, is constructed as follows: -

P	Q	~Q	P Λ ~Q	~(P Λ ~Q)
True	True	False	False	True
True	False	True	True	False
False	True	False	False	True
False	False	True	False	True

Tautologies & Contradictions

Some propositions contain *only true values in the last column of their truth tables* i.e. are true for any truth values of their variables. Such propositions are called **tautologies**. Similarly, a proposition P (P, Q …) is called a **contradiction** if it contains *only false values in the final column of its truth table* i.e. the statement is false for any truth values of its variables.

Example, the proposition "P and not P" (P Λ ~ P) is a contradiction as verified by constructing their truth tables as follows:

- P V ~ P is a Tautology

P	~P	P V ~ P
True	False	True
False	True	True

P Λ ~ P is a contradiction

P	~P	P Λ ~ P
True	False	False
False	True	False

The proposition (Q Λ R) V ~ (Q Λ R), which by principle of substitution must also be a tautology is proven using the truth table below: -

Q	R	Q Λ R	~ (Q Λ R)	(Q Λ R) V ~ (Q Λ R)
True	True	True	False	True
True	False	False	True	True
False	True	False	True	True
False	False	False	True	True

Review Questions

a) Construct truth tables for the following propositions: -

(i.) ~p Λ ~q

p	q	~p	~q	~p Λ ~q
T	T	F	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

(ii.)~ (~p v ~q)

p	q	~p	~q	(~p v ~q)	~ (~p v ~q)
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

b) Let p be “Roses are Red” and let q be “Violets are Blue”. Write up verbal sentences that describe each of the following statements: -

(i.) ~p

Roses are not red.

(ii.) ~ (p Λ q)

Its false that Roses are red and violets are blue

(iii.) p v q

Roses are Red or Violets are Blue

(iv.) q v ~p

Violets are Blue or Roses are not Red

(v.) ~p Λ ~q

Roses are not red and violets are not blue

(vi.) ~~q

Violets are blue

c) Construct truth tables for the following propositions: -

(i) p Λ ~q

p	q	~q	p Λ ~q
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

(ii) ~p Λ ~q

p	q	~p	~q	~p Λ ~q
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(iii) ~ (~p Λ q)

p	q	~p	~p Λ q	~ (~p Λ q)
T	T	F	F	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

d) Construct truth tables for the following statements

(i.) ~ (~p v ~q)

p	q	~p	~q	~p v ~q	~ (~p v ~q)
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

(ii.) ~ (~p V q)

p	q	~p	(~p V q)	~ (~p V q)
T	T	F	T	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	F

(iii.) p V (~p Λ q)

p	q	~p	(~p Λ q)	p V (~p Λ q)
T	T	F	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	F

e) Construct a truth table for each of the following statements:

(i.) q Λ (p→q)→p

p	q	p→q	(p→q)→p	q Λ (p→q)→p
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	F	F

(ii.) ~ (p Λ q) ↔ (~p V ~q)

p	q	~p	~q	(p Λ q)	~ (p Λ q)	~p V ~q	~ (p Λ q) ↔ (~p V ~q)
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

f) Conditional statements are those of the form "If p then q" They are the fundamental principles of logical reasoning. Given the conditional statement: - "If p implies q and q implies r then p implies r" verify this fact by showing that the following proposition is a tautology $[(p \rightarrow q) \Lambda (q \rightarrow r)] \rightarrow (p \rightarrow r)$.

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \rightarrow r$	$[(p \rightarrow q) \Lambda (q \rightarrow r)]$	$[(p \rightarrow q) \Lambda (q \rightarrow r)] \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

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