Automatic Control Systems

Lecture-2

Laplace Transforms and their Properties

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Laplace Transforms and their properties

Session Objectives

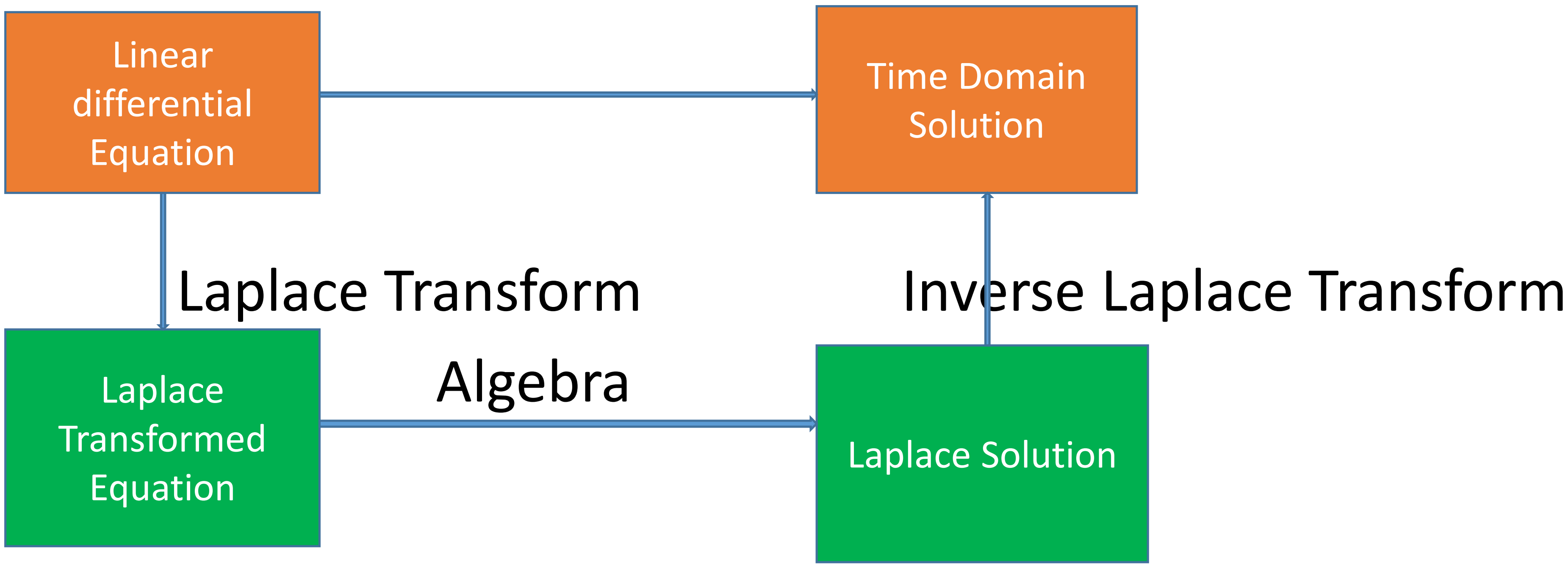
By the end of this session, learners will be able to:

- ➢ Define Laplace Transforms
- ➢ Find Laplace Transform of basic input signals used in control systems
- ➢ Find Inverse Laplace transform
- ➢ Perform partial fraction expansion
- ➢ Explain properties of Laplace transform

Definition of Laplace Transforms

- Laplace Transform is one of the mathematical tools used for solving Ordinary Linear Differential Equations.
- Laplace Transform converts differential into algebraic equations

Laplace Transformation



Mathematical Definition of Laplace Transform

$Laplace\ Transform\ of\ f(t)\ is\ defined\ as$

$$\text{L}[f(t)]=\text{F}(s)=\int_0^\infty f(t)\, e^{-st}\text{dt}$$

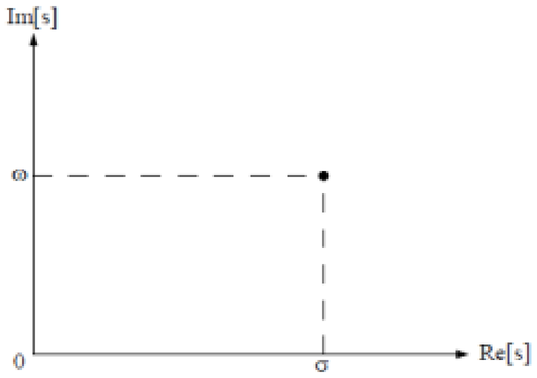
Convert time domain functions and operations into frequency domain

- $f(t) \Rightarrow F(s)\ (t \in R, s \in C)$
- Linear differential equations $\Rightarrow$ Algebraic expressions in complex plane

Katsuhiko Ogata (1997), Modern Control Engineering, Prentice Hall, page 17.

Laplace Transform Definition (cont'd)

The variable s is referred to as Laplace operator, which is a complex variable, s= $\sigma + j\omega$ where σ is a real part and ω is the imaginary part as shown in the figure below



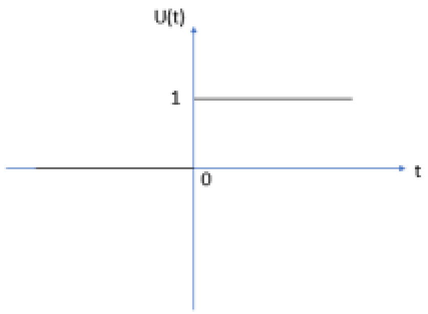
Graphical representation of Laplace operator (s-plane)

Laplace transform of Basic input signals

1. Unit step signal

Let f(t)be a unit step function that is defined to have a constant value of unity for t ≥ 0 and a zero for t < 0, namely, f(t)= Us (t).

$$f(t) = \mathrm{Us}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t \leq 0 \end{cases}$$

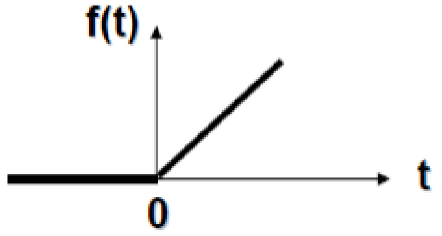


$$F(s) = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} Us(t)e^{-st}dt = \frac{1}{s}e^{-st}\Big\{\begin{matrix}\infty \\ 0\end{matrix} = -\frac{1}{s}[0-1] = \frac{1}{s}$$

2. Ramp Signal

A ramp function is defined as

$$f(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$



Graphical representation of a ramp function

$$F(s) = \int_0^\infty te^{-st}dt = -\frac{1}{s}[te^{-st}]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt = \frac{1}{s^2}$$

(Integration by parts)

3. Unit Impulse Signal

A unit impulse signal, (t) is defined as

$$\boldsymbol{\delta(t) = 0 \; for \; t \neq 0 \; and \; \int_{0^-}^{0^+} \delta(t)dt = 1}$$

Note that one of its important property:

$$\boldsymbol{F(s) = \int_0^\infty \delta(t)e^{-st}dt = e^{-s.0} = 1}$$

4. Exponential Function

Let $f(t) = e^{at}$ **for some a** $\in R$

$$L\{e^{at}\}$$
$$= \int_0^{\infty} e^{at} e^{-st} dt = \int_0^{\infty} e^{-(s-a)t} dt = -\frac{1}{s-a} e^{-(s-a)t} \Big\{\begin{matrix}\infty \\ 0\end{matrix} = \frac{1}{s-a}$$

Laplace Transforms

- Some commonly used Laplace Transforms and formulas
- Wide variety of function can be transformed
- Inverse Transform

$$\mathcal{L}^{-1}(F(s)) = f(t)$$

Katsuhiko Ogata (1997), Modern Control Engineering, Prentice Hall, page 22.

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$

Laplace Transforms (cont'd)

21. $\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$	22. $\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
23. $t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$	24. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
25. $u_c(t)=u(t-c)$	$\mathbf{e}^{-cs}$	26. $\delta(t-c)$	$\mathbf{e}^{-cs}$

LAPLACE TRANSFORMS

PARTIAL FRACTION EXPANSION

Definition—Partial fractions are several fractions whose sum equals a given fraction.

Purpose—Working with transforms requires breaking complex fractions into simpler fractions to allow use of tables of transforms

Partial Fraction Expansion

$$\frac{s+1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$\frac{s+1}{(s+2)(s+3)} = \frac{A(s+3)+B(s+2)}{(s+2)(s+3)}$$

$$A + B = 1 \qquad 3A + 2B = 1$$

$$\frac{s+1}{(s+2)(s+3)} = \frac{-1}{s+2} + \frac{2}{s+3}$$

- Expand into a term for each factor in the denominator
- Recombine RHS
- Equate terms in s and constants terms. Solve.
- Each term is in a form so that Inverse Laplace Transforms can be applied.

Katsuhiko Ogata (1997), Modern Control Engineering, Prentice Hall, page 35.

When the factors of the denominator are of the first degree but some are repeated, assume unkown numerator for each factor

If a term is present twice, make the fractions the corresponding term and its second power.

If a term is present three times, make the fractions the corresponding term and its second power and its third power.

Example

Find the partial fraction expansion of the following transfer function and its Inverse Laplace

$$G(s) = \frac{1}{s^2(s+1)}$$

Solution:

$G(s) = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{B}{s+1}$

Where

$A_2 = [s^2 G(s)] \Big|_{s=0} = \left[\frac{1}{s+1}\right] \Big|_{s=0} = 1$

$A_1 = \frac{d}{ds}\left[s^2 G(s)\right]\Big|_{s\,=\,0} = \frac{d}{ds}\left[\frac{1}{s+1}\right]\Big|_{s\,=\,0} = \left[\frac{-1}{(s+1)^2}\right]\Big|_{s\,=\,0} = -1$

And

$B = (s+1).G(s)\Big|_{s\,=\,-1} = 1$

Therefore,

$G(s) = \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$

The inverse Laplace of the given transfer function can be obtained in Laplace Transform Table

$$g(t) = L^{-1}\left(\frac{-1}{s}+\frac{1}{s^2}+\frac{1}{s+1}\right) = L^{-1}\left(\frac{-1}{s}\right) + L^{-1}\left(\frac{1}{s^2}\right) L^{-1}\left(\frac{1}{s+1}\right)$$

$$= -1 + t + e^{-t} \qquad t > 0$$

Exercises

Find the partial-fraction expansion of the following transfer function, and its inverse Laplace of the following transfer functions:

1. $G(s) = \frac{1}{s(s+1)^2(s+2)}$
2. $G(s) = \frac{s+5}{(s+1)\,(s+4)}$

Answers

1. Decomposition in partial fraction

$$G(s) = \frac{\frac{1}{3}}{s} - \frac{\frac{1}{4}}{s+1} - \frac{\frac{1}{2}}{(s+1)^2} - \frac{\frac{1}{12}}{s+3}$$

Inverse Laplace

$$g(t) = \frac{1}{3} - \frac{1}{4}e^{-t} - \frac{1}{2}te^{-t} - \frac{1}{12}e^{-3t} \qquad t > 0$$

2. Decomposition in partial fraction

$$G(s) = \frac{\frac{4}{3}}{s+1} - \frac{\frac{1}{3}}{s+4}$$

Inverse Laplace

$$g(t) = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t} \qquad t > 0$$

Properties of Laplace Transforms

- Linearity
- Scaling in time
- Time shift
- Frequency or s plane shift
- Integration
- Differentiation

Linearity

$$L\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} = \alpha_1 F_1(s) + \alpha_2 F_2(s)$$

Proof:

$L\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} =$

$\int_0^\infty [\alpha_1 f_1(t) + \alpha_2 f_2(t)] e^{-st} dt =$

$\alpha_1 \int_0^\infty f_1(t) e^{-st} dt + \alpha_2 \int_0^\infty f_2(t) e^{-st} dt =$

$\alpha_1 F_1(s) + \alpha_2 F_2(s)$

Scaling in time

$L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

Proof:

$L\{f(at)\} = \int_0^\infty f(at) e^{-st} dt$

Let $u = at,\ t = \frac{u}{a}, dt = \frac{1}{a} du$

$\frac{1}{a} \int_0^{\frac{\infty}{a}} f(u) e^{-\left(\frac{s}{a}\right)u} du\ =$

$\frac{1}{a} F\left(\frac{s}{a}\right)$

Time shift

$L\{f(t-t_0)u(t-t_0)\} = e^{-st_0}F(s)$

Proof:

$L\{f(t-t_0)u(t-t_0)\} = \int_0^\infty f(t-t_0)u(t-t_0)e^{-st}dt =$

$\int_{t_0}^\infty f(t-t_0)e^{-st}dt =$

Let $u = t - t_0, t = u + t_0$

$\int_0^{\infty - t_0} f(u)e^{-s(u+t_0)}du =$

$e^{-st_0}\int_0^\infty f(u)e^{-su}du = e^{-st_0}F(s)$

S-plane(frequency) shift

$L\{e^{-at}f(t)\} = F(s+a)$

Proof:

$L\{e^{-at}f(t)\} = \int_0^\infty e^{-at}f(t)e^{-st}dt =$

$\int_0^\infty f(t)e^{-(s+a)t}dt = F(s+a)$

Integration

$L\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s}$

Proof:

$$L\left\{\int_0^t f(t)dt\right\} = \int_0^\infty \left\{\int_0^t f(t)dt\right\} e^{-st}dt = \int_0^\infty \left\{\int_0^t f(t)dt\right\}\left(\frac{de^{-st}}{-s}\right)$$

$=\left(\frac{e^{-st}\int_0^t f(t)dt}{-s}\right)\begin{cases}\infty\\0\end{cases} + \frac{1}{s}\int_0^\infty e^{-st}d\left\{\int_0^t f(t)dt\right\}$

By fundamental theorem of calculus

$$\frac{d}{dt}\left\{\int_0^t f(t)dt\right\} = f(t) \rightarrow d\left\{\int_0^t f(t)dt\right\} = f(t)dt$$

The Laplace Transform then becomes

$= \frac{1}{s}\int_0^\infty e^{-st}f(t)dt = \frac{F(s)}{s}$

Differentiation

First order differential Equation

$$L\left\{\frac{df(t)}{dt}\right\} = SF(s) - f(0)$$

With $f(0)$ *is an initial condition*.

Proof:

$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt$

Now we use integration by parts

Let

$u = e^{-st}$

$du = -se^{-st}dt$

$dv = f'(t)dt$

$v = f(t)$

Using these substitution in the integral by parts formula, we get

$$L\{f'(t)\} = [e^{-st}f(t)]\begin{cases}\infty \\ 0\end{cases} - \int_0^\infty f(t)(-se^{-st})dt$$

$$=\left[\frac{f(t)}{e^{-st}}\right]\begin{cases}\infty \\ 0\end{cases} + s\int_0^\infty e^{-st}f(t)dt$$

$$=\left[\frac{f(t)}{e^{-}}\right]\begin{cases}\infty \\ 0\end{cases} + SL\{f\ (t)\}$$

If we the apply the limits, we get:

$$L\{f'(t)\} = \left[\frac{f(\infty)}{e^\infty} - \frac{f(0)}{e^0}\right] + sL\{f(t)\}$$

$$L\{f'(t)\} = sL\{f(t)\} - \mathrm{f}(0)$$

Differentiation(cont'd)

Second order differential Equation

$$L\left\{\frac{d^2f(t)}{dt^2}\right\} = S^2F(s) - Sf(0) - f'(0)$$

With $f(0)\ and\ f'(0)$ are initial conditions

We can deduce that the general the general Laplace Transform for derivative of $f^{(n)}$is:

$$L\left(\frac{d^nf}{dt^n}\right) = s^nLf(t) - s^{n-1}f(0) - s^{n-2}f(0) - \cdots - f^{n-1}\ (0)$$

References

1. Katsuhiko Ogata (1997), Modern Control Engineering, Prentice Hall.
2. Benjamin C. Kuo (1975), Automatic Control Systems, 3rd Edition, Prentice Hall.
3. Smarajit Gosh (2007), Control systems, Pearson Education.

THANK YOU FOR YOUR ATTENTION!!