Automatic Control Systems

Lecture-3

Solving Differential Equations Using Laplace Transforms

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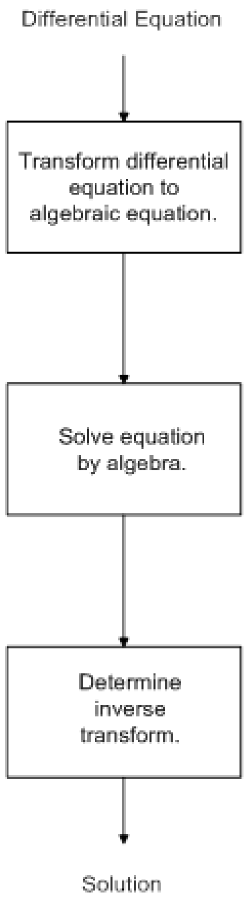
Laplace Transforms and their properties

Session Objectives

By the end of this session, learners will be able to:

- Explain steps to follow while solving differential Equations
- Solve 1st Order Differential Equations using Laplace Transforms
- Solve 2nd Order Differential Equations using Laplace Transforms.

Steps Involved in Using Laplace Transform Methods



Recall

The main tools we need from the last lecture

$$L\{f(t)\} = F(s)$$

$$L\left\{\frac{df(t)}{dt}\right\} = SF(s) - f(0)$$

$$L\left\{\frac{d^2f(t)}{dt^2}\right\} = S^2F(s) - Sf(0) - f'(0)$$

$$L\left(\frac{d^nf}{dt^n}\right) = s^nLf(t) - s^{n-1}f(0) - s^{n-2}f(0) - \cdots - f^{n-1}(0)$$

Benjamin C. Kuo (1975), Automatic Control Systems, 3rd Edition, Prentice Hall.

Inverse Laplace Transform by Identification

When a differential equation is solved by Laplace transforms, the solution is obtained as a function of variable s. The inverse transform must be formed in order to determine the time response. The simplest forms are those that can be recognized within the tables and few of those will now be recognized.

K. Ogata (1997), Modern Control Engineering, 3rd Edition, Prentice Hall, page 44.

Solved Examples

Example 1:

Solve using the Laplace Transform

$y' - y = e^{3t} \quad\quad y(0) = 2$

Firstly, apply Laplace on both sides

Left hand side

$L(y' - y) = L(y') - L(y)$

$= SY(s) - y(0) - Y(s)$

$= [S - 1]Y(s) - 2$

Right hand side

$$L(e^{3t}) = \frac{1}{S-3}$$

Taking both the left hand and right hand sides give

$$[S-1]Y(s) - 2 = \frac{1}{S-3}$$

$$Y(s) = \left(\frac{1}{S-3} + 2\right)\left(\frac{1}{S-1}\right)$$

$$Y(s) = \frac{2S-5}{(S-3)(S-1)}$$

Perform Partial fraction expansion

$$Y(s) = \frac{2S-5}{(S-3)(S-1)} = \frac{A}{S-3} + \frac{B}{S-1}$$

$$A = \frac{1}{2}$$

$$B = \frac{3}{2}$$

$$Y(s) = \frac{\frac{1}{2}}{S-3} + \frac{\frac{3}{2}}{S-1}$$

Inverse Laplace is obtained by using Laplace Table

$$y(t) = \frac{1}{2}(e^{3t} + 3e^{t})$$

Example 2.

Solve $y'' - 3y' + 2y = e^{3t} \quad y'(0) = 0 \;\; y(0) = 1$

Apply Laplace Transform on both sides

Left hand side

$L(y'' - 3y' + 2y)$

$= S^2Y(s) - Sy(0) - y'(0) - 3[SY(s) - y(0)] + 2Y(s)$

$=[S^2 - 3S + 2]Y(s) - S + 3$

Right hand side

$L(e^{3t}) = \frac{1}{S-3}$

Taking both the left hand and right hand sides give

$[S^2 - 3S + 2]Y(s) - S + 3 = \frac{1}{S-3}$

$Y(s) = \frac{S^2-6S+10}{(S-3)(S-2)(S-1)}$

Perform partial fraction expansion

$$Y(s) = \frac{S^2-6S+10}{(S-3)(S-2)(S-1)} = \frac{A}{S-3} + \frac{B}{S-2} + \frac{C}{S-1}$$

$$A = \frac{1}{2}$$

$$B = -2$$

$$C = \frac{5}{2}$$

$$Y(s) = \frac{\frac{1}{2}}{S-3} - \frac{2}{S-2} + \frac{\frac{1}{2}}{S-1}$$

Inverse Laplace is obtained by using Laplace Table

$$y(t) = \frac{1}{2}(e^{3t} - 4e^{2t} + 5e^{t})$$

Example 3.

Solve $y'' - 10y' + 9y = 5t \quad y(0) = -1 \quad y'(0) = 2$

Applying the Laplace Transform on both sides, we get

Left Hand Side

$L(y'' - 10y' + 9y)$

$= S^2Y(s) - Sy(0) - y'(0) - 10[\mathrm{SY(s) - y(0)}] + 9\mathrm{Y(s)}$

$=S^2Y(s) - Y(s)[10S - 9] + S - 12$

Right Hand Side

$$\boldsymbol{L(5t) = \frac{5}{S^2}}$$

Taking both the left hand and right hand sides give

$$S^2Y(s) - Y(s)[10S - 9] + S - 12 = \frac{5}{S^2}$$

$$\boldsymbol{Y(s) = \frac{-S^3+12S^2+5}{S^2(S^2-10S+9)}}$$

$$\boldsymbol{Y(s) = \frac{-S^3+12S^2+5}{S^2(S-9)(S-1)}}$$

Perform Partial Fraction Expansion

$$Y(s) = \frac{A_1}{S} + \frac{A_2}{S^2} + \frac{B}{S-9} + \frac{C}{S-1}$$

$$A_1 = \frac{50}{81}$$

$$A_2 = \frac{5}{9}$$

$$B = \frac{31}{81}$$

$$C = -2$$

$$Y(s) = \frac{\frac{50}{81}}{S} + \frac{\frac{5}{9}}{S^2} + \frac{\frac{31}{81}}{S-9} - \frac{2}{S-1}$$

Inverse Laplace is obtained by using Laplace Table

$$y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{9t} - 2e^{t}$$

Example 4.

Solve $y'' - 6y' + 15y = 2sin3t \quad y(0) = -1 \quad y'(0) = -4$

Apply Laplace Transform on both sides, we get

Left Hand Side

$L(y'' - 6y' + 15y)$

$= S^2Y(s) - Sy(0) - y'(0) - 6SY(s) + 6y(0) + 15Y(s)$

$= Y(s)[S^2 - 6S + 15] + S - 2$

Right Hand Side

$$L(2sin3t) = 2\frac{3}{S^2+9}$$

Taking both the left hand and right hand sides give

$$Y(s)[S^2 - 6S + 15] + S - 2 = \frac{6}{S^2+9}$$

$$Y(s)[S^2 - 6S + 15] = \frac{6}{S^2+9} - S + 2$$

$$Y(s)[S^2 - 6S + 15] = \frac{-S^3+2S^2-9S+24}{S^2+9}$$

$$Y(s) = \frac{-S^3+2S^2-9S+24}{(S^2+9)(S^2-6S+15)} = \frac{AS+B}{S^2+9} + \frac{CS+D}{S^2-6S+15}$$

To find the constants, we need to simplify the expression on the right (find the common denominator) and equate the coefficients at the equal powers:

S^3: $A + C = -1 \ldots\ldots\ldots\ldots\ldots.(1)$

S^2: $-6A + B + D = 2 \ldots\ldots\ldots(2)$

S^1: $15A - 6B + 9C = -9 \ldots.(3)$

S^0: $15B + 9D = 24 \ldots\ldots\ldots\ldots.(4)$

From equation (1) $C = -1 - A \ldots\ldots\ldots..(5)$

From equation (4) $D = \frac{24-15B}{9} \ldots\ldots\ldots\ldots\ldots(6)$

Putting equation (5) into equation (3), we get

$$15A - 6B + 9(-1 - A) = -9$$

$$6A - 6B = 0$$

$$A - B = 0 \ldots\ldots\ldots\ldots.(7)$$

Putting equation (6) into equation (2), we get

$$-6A + B + \frac{(24-15B)}{9} = 2$$

$$-9A - B = -1 \ldots \ldots \ldots (8)$$

Solving equation (7) and equation (8), we get

$$A = \frac{1}{10}$$

$$C = -1 - \frac{1}{10} = -\frac{11}{10}$$

$$D = \frac{24 - \frac{15}{10}}{9} = \frac{240 - 15}{9} = \frac{225}{9} = \frac{5}{2}$$

By solution

$$A = \frac{1}{10}$$

$$B = \frac{1}{10}$$

$$C = -\frac{11}{10}$$

$$D = \frac{5}{2}$$

Hence we get

$$Y(s) = \frac{1}{10}\left(\frac{S+1}{S^2+9} + \frac{-11S+25}{S^2-6S+15}\right)$$

We need to find Inverse Laplace

Starting from the first term

$$L^{-1}\left(\frac{S+1}{S^2+9}\right) = L^{-1}\left(\frac{S}{S^2+9} + \frac{1}{S^2+9}\right) = L^{-1}\left(\frac{S}{S^2+9}\right) + L^{-1}\left(\frac{1}{S^2+9}\right)$$

$$L^{-1}\left(\frac{S}{S^2+9}\right) + \frac{1}{3}L^{-1}\left(\frac{3}{S^2+9}\right)$$

$$= cos3t + \frac{1}{3}sin3t$$

The second term is slightly more complex

Remember that we can always **add** and **subtract** the same expression and **multiply** and **divide** by the same expression different from zero

$$\frac{-11S+25}{S^2-6S+15} = \frac{-11S+25}{(s-3)^2+6} = \frac{-11(S-3)-8}{(s-3)^2+6}$$

$$= \frac{-11(S-3)}{(s-3)^2+6} - \frac{8}{\sqrt{6}}\frac{\sqrt{6}}{(s-3)^2+6}$$

Now

$$L^{-1}\left(\frac{-11S+25}{S^2-6S+15}\right) = -11e^{3t}cos\sqrt{6}t - \frac{8}{\sqrt{6}}e^{3t}sin\sqrt{6}t$$

Hence the final answer is:

$$y(t) = \frac{1}{10}\left(cos3t + \frac{1}{3}sin3t - 11e^{3t}cos\sqrt{6}t - \frac{8}{\sqrt{6}}e^{3t}sin\sqrt{6}t\right)$$

References

1. Benjamin C. Kuo (1975), Automatic Control Systems, 3rd Edition, Prentice Hall.
2. K. Ogata (1997), Modern Control Engineering, 3rd Edition, Prentice Hall.
3. Smarajit Gosh (2007), Control systems, Pearson Education.

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