Automatic Control Systems

Lecture-5

Modeling Mechanical Systems

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Modeling Mechanical Systems

Session Objectives

By the end of this session, learners will be able to:

- ➢Definition of Mechanical Systems
- ➢Types of Mechanical Systems
- ➢Parameters involved in Mechanical Systems
- ➢Find the transfer function of a Mechanical System
- ➢Determine the relationship between Electrical Systems and Mechanical Systems.

Definition

A mechanical system is a set of physical components that convert input motion (force) into desired output motion and force.

Types of Mechanical Systems

Based on the types of motions we have:

- *Translational Mechanical Systems*
- *Rotational Mechanical Systems*

Translational Mechanical system

The motion is translational motion

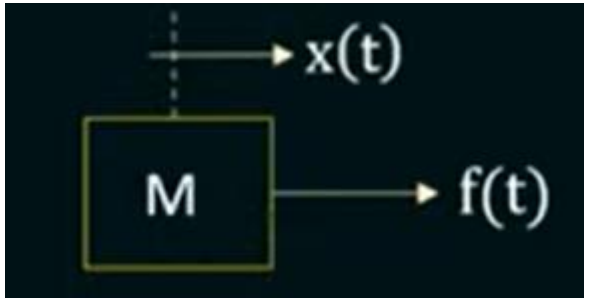
- The motion is in a straight line
- The equation governing such motion are Newton’s Law of motion

Parameters of Translational Motion

- Dispalcement: Change in position of an object w.r.t time $\{x(t)\}$ $[m]$
- Velocity: Rate of change of displacement w.r.t time $\left\{v = \frac{dx}{dt}\right\}$ $[m/s]$
- Acceleration: Rate of change of velocity w.r.t time $\left\{a = \frac{dv}{dt} =\right.$

Components or Elements of Translational Mechanical System

1. **Mass:** This a property of the system itself which stores the kinetic energy.



Displacement of mass always takes place in the direction of force applied.

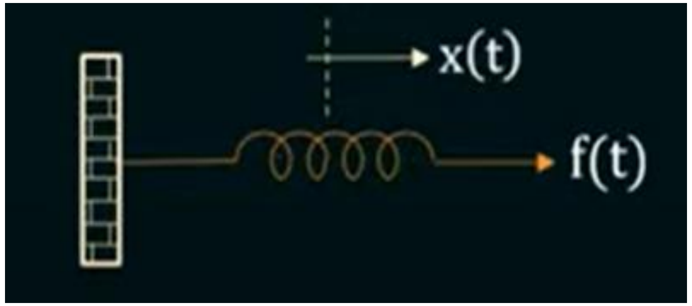
$f(t) \propto a \qquad f(t) = M.a \qquad [\mathrm{Kg}]$

2. Dashpot: Motion is opposed by friction

$$f_v(t) \propto \frac{dx}{dt}$$

$$f_c(t) = B\frac{dx}{dt}, \quad \mathrm{B} = \text{Frictional Consatant} \quad \left[\mathrm{N}.\frac{\mathrm{s}}{\mathrm{m}}\right]$$

3. Spring: A spring is an elastic object that stores potential energy

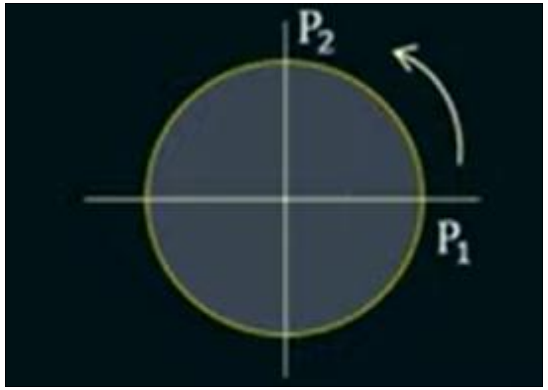


$$f(t) \propto x(t)$$

$$f(t) = K.x(t), K \text{ is the spring constant} \quad \left[\frac{N}{m}\right]$$

Rotational Mechanical System

Rotational motion is a circular motion of a rigid body about a fixed axis.



Parameters of Rotational Motion

1. Angular displacement: Angle in radians through which rigid body rotate about a fixed axis (θ) SI unit: Radians

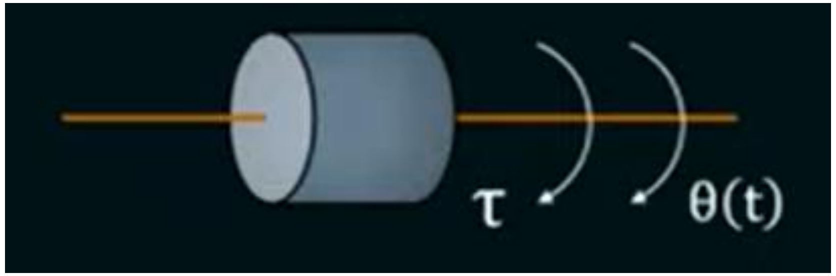
2. Angular velocity: Rate of change of angular position $\Big(\omega =$

Parameters of Rotational Motion(cont'd)

3. Angular acceleration: Rate of change of angular velocity $\left(\alpha = \frac{d\omega}{dt} = \right.$

Components (elements) in Rotational Mechanical System

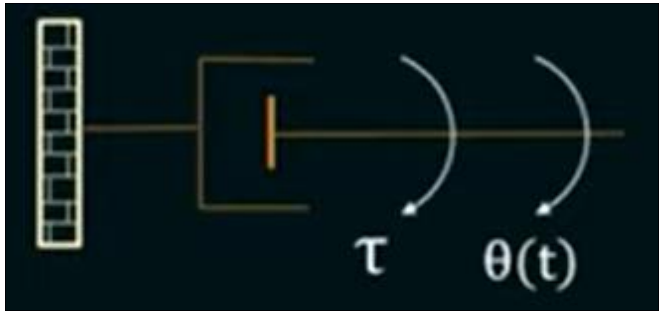
1. **Moment of Inertia**: It is property of rigid body by virtue of which it opposes the change in its angular position or angular velocity $[J]$



$$\tau \propto \alpha, \qquad \tau = J.\alpha$$

$$\tau = J.\frac{d^2\theta(t)}{dt^2} \rightarrow \tau(s) = J.S^2\theta(s)$$

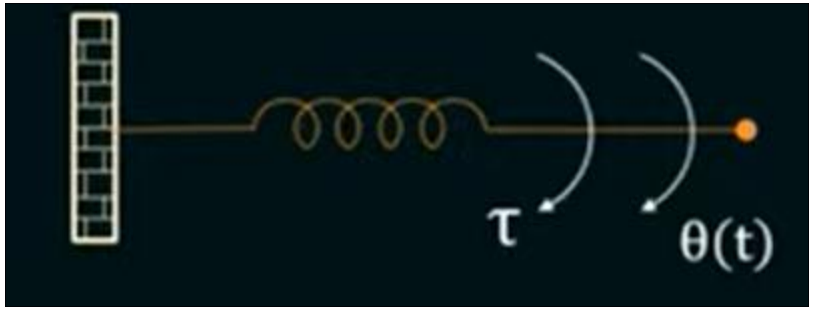
2. **Dashpot**:



$$\tau \propto \frac{d\theta(t)}{dt}, \qquad \tau = B.\frac{d\theta(t)}{dt} \qquad \left[N.m.\frac{s}{rad}\right]$$

$$\tau(s) = B.S\theta(s)$$

3. **Spring**:



$\tau \propto \theta(t), \qquad \tau = K.\theta(t)$

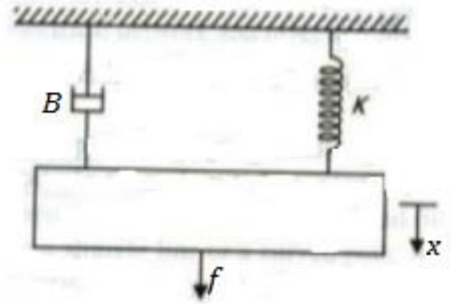
$\tau(s) = K.\theta(s) \qquad [N.\frac{m}{rad}]$

Comparison of Translational and Rotational Mechanical Systems

Translational	Rotational
1. Mass [M]	Moment of Inertia [J]
2. Force [F]	Torque τ
3. Spring consatant [K]	Torsional spring constant [K]
4. Frictional constant [B]	Torsional frictional constant [B]
5. Displacement [X]	Angular displacement [θ]
6. Velocity [$v = \frac{dx}{dt}$]	Angular velocity [$\omega = \frac{d\theta}{dt}$]
7. Acceleration [$a = \frac{dv}{dt}$]	Angular acceleration [$\alpha = \frac{d\omega}{dt}$]

Mathematical Model of a Mechanical System

Determine a mathematical model of a system shown in the following figure



The force f causes the displacement of mass M, this is opposed by spring as well as by dashpot.

Mathematical Model of a Mechanical System (cont'd)

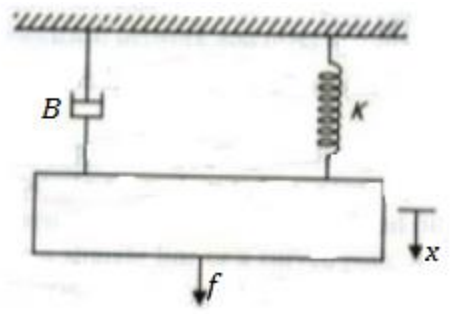
The forces $K.x$ and $B.\frac{dx}{dt}$ will oppose f

$$M\frac{d^2x}{dt^2} = f - B.\frac{dx}{dt} - K.x$$

$$M\frac{d^2x}{dt^2} + B.\frac{dx}{dt} + K.x = f$$

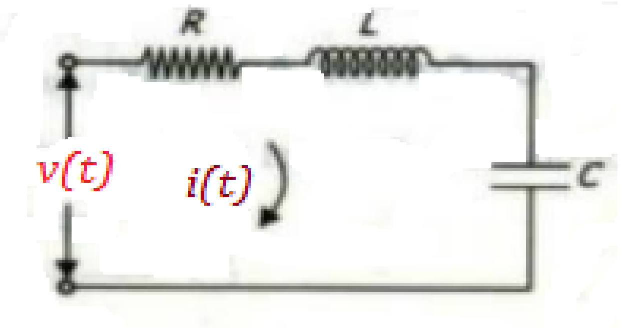
Analogous Systems

Translational System



$$M\frac{d^2x}{dt^2} + B.\frac{dx}{dt} + K.x = f$$

Electrical System



$$v(t) = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q$$

Analogous Systems (cont'd)

There are two methods to get analogous systems.

These are:

(i) Force-Voltage (f-v) analogy and

(ii) Force-current (f-i) analogy

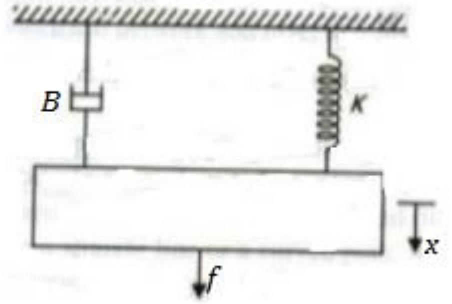
Smarajit Gosh (2007), Control systems, Pearson Education, pages 122-123

Force-Voltage (f-v) Analogy

Mechanical system

Input: Force

Output: Displacement

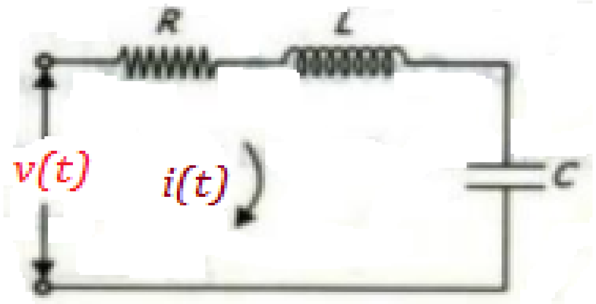


$$M\frac{d^2x}{dt^2} + B.\frac{dx}{dt} + K.x = f$$

Electrical system

Input: Voltage

Output: Current



$$v(t) = Ri(t) + L\frac{di}{dt} + \frac{1}{C}\int idt$$

Using Laplace Transform on the mechanical system equation, we get

$$MS^2X(s) + BSX(s) + KX(s) = F(s) \ldots\ldots\ldots..(1)$$

Using Laplace Transform on the electrical system equation, we get

$$RI(s) + SLI(s) + \frac{I(s)}{CS} = V(s) \ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

Taking $i = \frac{dq}{dt}$ $\quad thus \quad I(s) = SQ(s) \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3)$

Putting equation (3) into equation (2), we get

$$V(s) = S^2LQ(s) + RSQ(s) + \frac{1}{C}Q(s) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Comparing F(s) and V(s)

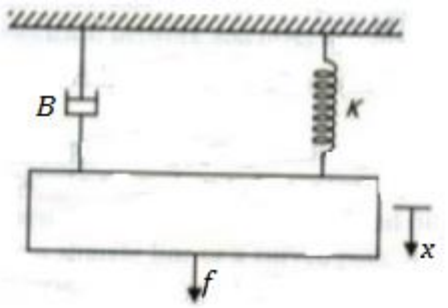
Mechanical	**Electrical**
M	L
B	R
K	1/C
X	Q

Force-Current (f-i) Analogy

Mechanical system

Input: Force

Output: Displacement

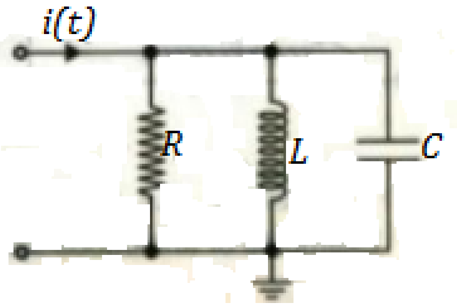


$$M\frac{d^2x}{dt^2} + B.\frac{dx}{dt} + K.x = f$$

Electrical system

Input: Current

Output: Flux



$using\ nodal\ analysis$

$$i(t) = i_R + i_L + i_C$$

Using Laplace Transform on the mechanical system equation, we get

$$MS^2X(s) + BSX(s) + KX(s) = F(s) \ldots\ldots\ldots..(1)$$

And

$i_{R=\frac{v}{R}}, \quad i_L = \frac{1}{L}\int v dt, \quad i_C = C\frac{dv}{dt}$

Thus , $i_{t=\frac{v}{R}, +\frac{1}{L}\int v dt + C\frac{dv}{dt}}$

By applying Laplace Transform, we get

$$I(s) = \frac{V(s)}{R} + \frac{1}{L}\frac{V(s)}{S} + SCV(s) \ \ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

Taking v(t) $= \frac{d\emptyset}{dt}$ $\quad thus \quad V(s) = S\emptyset(s) \ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (3)$

Putting equation (3) into equation (2), we get

$$I(s) = S^2C\emptyset(s) + \frac{1}{R}S\emptyset(s) + \frac{1}{L}\emptyset(s) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Comparing F(s) and V(s)

Mechanical	**Electrical**
M	C
B	1/R
K	1/L
X	ϕ

References

1. Smarajit Gosh (2007), Control systems, Pearson Education
2. Manirakiza and Kanyarwanda (2020), ELT 303Lecture Note, IPRC Gishari, pages 26-31.
3. K. Ogata (1997), Modern Control Engineering, 3rd Edition, Prentice Hall.

THANK YOU FOR YOUR KIND ATTENTION