Mathematics for Science

Lecture 7

Decomposing Rational Functions: CASE III and IV

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Introduction to Lecture 7

This lecture is a continuation to Lecture 6. It will introduce the two other cases i.e. CASE III and CASE IV. These are cases where the denominator is a product of quadratic factors.

Intended learning outcomes

At the end of this lecture you will be able to;

(i) Differentiate between CASE III and CASE IV.

(ii) Decompose rational functions CASES III and IV.

Further Readings

This lecture 7 notes should be complemented with relevant topics in (Kahenya, 2017; Murray & Robert, 2009; Stewart, 2012). Murray & Robert (2009) have more additional details on rational functions especially on the asymptotes and graphing of rational functions while Stewart (2012) has application of decomposition of rational functions in integration.

Definition 1: (Rules of decomposing a proper fraction)

In lecture 6 we dealt with Rules 1 and 2. In this lecture we shall deal with Rules 3 and 4. i.e.,

Rule 3: If the denominator can be factorized into quadratic factors that occurs once, then we have what we call CASE III.

Rule 4: If the denominator can factorize into quadratic factors with some being repeated then we call this CASE IV.

Cases III and IV is where the factors of the denominator are quadratic factors and in some cases we also have some linear factors.

CASE III: The denominator $\mathbf{q(x)}$ is a product of unique (non-repeated) quadratic factor(s)

Given the rational fraction $\frac{p(x)}{q(x)}$ where $q(x)$ has the factor $ax^2 + bx + c$ where $b^2 - 4ac < 0$ (in additional to maybe some linear factors), then the fraction can be decomposed into;

$$\frac{Ax + B}{ax^2 + bx + c} \text{ with constants A and B to be determined.}$$

Example 1: Decompose into partial fractions

$$\frac{2x^2 - x + 6}{x^3 + 6x}$$

Solution: The denominator can be factorized into a linear factor x and an irreducible quadratic factor $x^2 + 6$ i.e.

$$\frac{2x^2 - x + 6}{x^3 + 6x} = \frac{2x^2 - x + 6}{x(x^2 + 6)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 6}$$

$$\Rightarrow \frac{2x^3 - x + 6}{x(x^2 + 6)} = \frac{A(x^2 + 6) + (Bx + C)x}{x(x^2 + 6)}$$

Hence we get; $2x^3 - x + 6 \equiv A(x^2 + 6) + x(Bx + C)$

$$2x^2 - x + 6 \equiv Ax^2 + 6A + Bx^2 + Cx = (A + B)x^2 + Cx + 6A$$

Equating the two sides to get; $6 = 6A \therefore A = 1 \Rightarrow C = -1$

$$A + B = 2 \Rightarrow B = 1$$

Therefore; $\frac{2x^2 - x + 6}{x^3 + 6x} = \frac{1}{x} + \frac{x - 1}{x^2 + 6} = \frac{1}{x} + \frac{x}{x^2 + 6} - \frac{1}{x^2 + 6}$

Example 2: Decompose into partial fractions;

$$\frac{2x^2 + 3x + 1}{(x - 7)(x^2 + 3)}$$

Solution: The denominator has two factors: a linear factor $(x - 7)$ and a quadratic factor $(x^2 + 3)$. Therefore we have;

$$\frac{2x^2 + 3x + 1}{(x - 7)(x^2 + 3)} = \frac{A}{x - 7} + \frac{Bx + C}{x^2 + 3} = \frac{A(x^2 + 3) + (Bx + C)(x - 7)}{(x - 7)(x^2 + 3)}$$

$$\Rightarrow 2x^2 + 3x + 1 \equiv A(x^2 + 3) + (Bx + C)(x - 7)$$

Expanding the RHS to get;

$$2x^2 + 3x + 1 = Ax^2 + 3A + Bx^2 - 7Bx + Cx - 7C$$

$$= (A + B)x^2 + (C - 7B)x + (3A - 7C)$$

Equating the two sides we get;

$$A + B = 2 \cdots (i)$$

$$C - 7B = 3 \cdots (ii)$$

$$3A - 7C = 1 \cdots (iii)$$

From (i) $A = 2 - B$. Replacing A in (ii) to get $3(2 - B) - 7C = 1 \Rightarrow 6 - 3B - 7C = 1$

Hence $3B + 7C = 5 \cdots (iv)$

But from (iii) $C = 3 + 7B$. Therefore equation (iv) becomes;

$$3B + 7(3 + 7B) = 5$$

$$3B + 21 + 49B = 5 \Rightarrow 52B = -16 \therefore B = -\frac{4}{13}$$

Then $C = 3 + 7B = 3 - \frac{28}{13} = \frac{11}{13}$ and $A = 2 - B = 2 + \frac{4}{13} = \frac{30}{13}$

Hence our fraction can be decomposed into;

$$\frac{2x^2 + 3x + 1}{(x - 7)(x^2 + 3)} = \frac{30}{13(x - 7)} + \frac{-\frac{4}{13}x + \frac{11}{13}}{x^2 + 3} = \frac{30}{13(x - 7)} - \frac{4x}{13(x^2 + 3)} + \frac{11}{13(x^2 + 3)}$$

$$\frac{2x^2 + 3x + 1}{(x - 7)(x^2 + 3)} = \frac{30}{13(x - 7)} - \frac{4x}{13(x^2 + 3)} + \frac{11}{13(x^2 + 3)}$$

Example 3: Decompose into partial fractions

$$\frac{3x^2 + 5x + 7}{3x^2 + 2x + 1}$$

Solution: Note that this is an improper fraction. We need first to divide to get;

$$\begin{array}{r|r} & 1 \\ \cline{2-2} 3x^2 + 2x + 1) & 3x^2 + 5x + 7 \\ & -3x^2 - 2x - 1 \\ \cline{2-2} & 3x + 6 \end{array}$$

That is;

$$\frac{3x^2 + 5x + 7}{3x^2 + 2x + 1} = 1 + \frac{(3x + 6)}{3x^2 + 2x + 1} = 1 + \frac{3x}{3x^2 + 2x + 1} + \frac{6}{3x^2 + 2x + 1}$$

Example 4: Decompose into partial fractions

$$\frac{7x^2 + 2x + 4}{(x^2 + 3)(x^2 - 7)}$$

Solution: The denominator is a product of two quadratic factors that are irreducible in $\mathbb{R}$.

It can then be decomposed into;

$$\frac{7x^2 + 2x + 4}{(x^2 + 3)(x^2 - 7)} = \frac{Ax + B}{x^2 + 3} + \frac{Cx + D}{x^2 - 7}$$

$$\Rightarrow \frac{7x^2 + 2x + 4}{(x^2 + 3)(x^2 - 7)} = \frac{Ax + B}{x^2 + 3} + \frac{Cx + D}{x^2 - 7} = \frac{(Ax + B)(x^2 - 7) + (x^2 + 3)(Cx + D)}{(x^2 + 3)(x^2 - 7)}$$

$$\therefore 7x^2 + 2x + 4 \equiv (Ax + B)(x^2 - 7) + (x^2 + 3)(Cx + D)$$

$$7x^2 + 2x + 4 \equiv (C + A)x^3 + (D + B)x^2 + (3C - 7A)x + 3D - 7B \cdots (i)$$

From (i)

$$3D - 7B = 4 \cdots (*)$$

$$3C - 7A = 2 \cdots (**)$$

$$D + B = 7 \cdots (***)$$

$$C + A = 0 \cdots (****)$$

From (**) and (****); $3C - 7(-C) = 2 \Rightarrow 3C + 7C = 2 \therefore C = \frac{1}{5}$ and $A = -\frac{1}{5}$

From (*) and (***); $3D - 7(7 - D) = 3D - 49 + 7D = 4 \Rightarrow 10D = 53 \therefore D = 5.3$ and $B = 1.7$

Our fraction becomes;

$$\frac{7x^2 + 2x + 4}{(x^2 + 3)(x^2 - 7)} = \frac{-\frac{1}{5}x + \frac{17}{10}}{x^2 + 3} + \frac{\frac{1}{5}x + \frac{53}{10}}{x^2 - 7} = \frac{17 - 2x}{10(x^2 + 3)} + \frac{2x + 53}{10(x^2 - 7)}$$

$$\frac{7x^2 + 2x + 4}{(x^2 + 3)(x^2 - 7)} = \frac{17}{10(x^2 + 3)} - \frac{x}{5(x^2 + 3)} + \frac{x}{5(x^2 - 7)} + \frac{53}{10(x^2 - 7)}$$

CASE IV: The denominator $\mathbf{q(x)}$ contains a repeated irreducible quadratic factor(s)

In case the denominator $q(x)$ has the irreducible factor $(ax^2 + bx + c)^r$ where $b^2 - 4ac < 0$ then we have;

$$\frac{p(x)}{q(x)} = \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

Example 1: Decompose into partial fractions the rational fraction;

$$\frac{x^3 + 7x^2 + 5}{x(x^2 + 7)^2}$$

Solution: The denominator consists of a linear factor x and a repeated quadratic factor $(x^2 + 7)$

$$\frac{x^3 + 7x^2 + 5}{x(x^2 + 7)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 7} + \frac{Dx + E}{(x^2 + 7)^2}$$

Simplifying the RHS to get;

$$\frac{x^3 + 7x^2 + 5}{x(x^2 + 7)^2} = \frac{A(x^2 + 7)^2 + x(Bx + C)(x^2 + 7) + x(Dx + E)}{x(x^2 + 7)^2}$$

$$\Rightarrow 7x^2 + 5 \equiv A(x^2 + 7)^2 + x(Bx + C)(x^2 + 7) + x(Dx + E)$$

Next we expand and simplify the RHS to get;

$$x^3 + 7x^2 + 5 \equiv (A + B)x^4 + Cx^3 + (14A + 7B + D)x^2 + (7C + E)x + 49A$$

Equating the two sides we get the following equations; $49A = 5 \therefore A = \frac{5}{49} \cdots \text{(i)}$

$$7C + E = 0 \cdots \text{(ii)}$$

$$14A + 7B + D = 7 \Rightarrow 7B + D = 7 - \frac{10}{7} = \frac{39}{7} \therefore\ 7B + D = \frac{39}{7} \cdots \text{(iii)}$$

$$C = 1 \cdots \text{(iv)}\ \Rightarrow E = -7 \text{ (from equation (ii))}$$

$$A + B = 0 => B = -A = -\frac{5}{49} \text{ (from equation (i))}$$

From equation (ii) $D = \frac{39}{7} - 7B = \frac{39}{7} + \frac{5}{7} = \frac{44}{7}$

Therefore our constants $\{A, B, C, D, E\} = \left\{\frac{5}{49}, -\frac{5}{49}, 1, \frac{44}{7}, -7\right\}$

Therefore decomposition our fraction into partial fractions we get;

$$\frac{x^3 + 7x^2 + 5}{x(x^2 + 7)^2} = \frac{5}{49x} + \frac{-\frac{5}{49}x + 1}{x^2 + 7} + \frac{\frac{44}{7}x - 7}{(x^2 + 7)^2}$$

$$\frac{x^3 + 7x^2 + 5}{x(x^2 + 7)^2} = \frac{5}{49x} + \frac{1}{(x^2 + 7)} - \frac{5x}{49(x^2 + 7)} + \frac{44x}{7(x^2 + 7)^2} - \frac{7}{(x^2 + 7)^2}$$

Example 2: Decompose into partial fractions

$$\frac{x^2+x+2}{x^2+x+7}$$

Solution:

$$\frac{x^2+x+2}{x^2+x+7}=\frac{Ax+B}{x^2+x+7}$$

$$\Rightarrow x^2+x+2=Ax+B$$

Comparing the two sides we get;

$$B=2 \text{ and } A=1$$

Hence we have;

$$\frac{x^2+x+2}{x^2+x+7}=\frac{x+2}{x^2+x+7} \text{ INVALID}$$

Note that the numerator and the denominator of the rational function are of the same degree.

We use long division to get;

$$\begin{array}{r|r} & 1 \\ x^2+x+7 & x^2+x+2 \\ & -x^2-x-7 \\ \hline & -5 \end{array}$$

$$\therefore \quad \frac{x^2+x+2}{x^2+x+7}=1-\frac{5}{x^2+x+7}$$

Example 3: Decompose into partial fractions

$$\frac{2x^3+3x^2+x+1}{(x^2+x+3)(x^2-7)^2}$$

Solution: The denominator consists of two unique quadratic factors (x^2+x+3) and (x^2-7), with (x^2-7) repeated twice.

The rational fraction can be decomposed into;

$$\frac{2x^3+3x^2+x+1}{(x^2+x+3)(x^2-7)^2}=\frac{Ax+B}{x^2+x+3}+\frac{Cx+D}{x^2-7}+\frac{Ex+F}{(x^2-7)^2}$$

Simplifying the RHS to get;

$$\frac{2x^3+3x^2+x+1}{(x^2+x+3)(x^2-7)^2}=\frac{(Ax+B)(x^2-7)^2+(Cx+D)(x^2+x+3)(x^2-7)+(Ex+F)(x^2+x+3)}{(x^2+x+3)(x^2-7)^2}$$

$$\Rightarrow 2x^3+3x^2+x+1\equiv(Ax+B)(x^2-7)^2+(Cx+D)(x^2+x+3)(x^2-7)+(Ex+F)(x^2+x+3)$$

Expanding and simplifying the RHS to get the polynomial;

$$=(C+A)x^5+(D+C+B)x^4+(E+D-4C-14A)x^3+(F+E-4D-7C-14B)x^2+(F+3E-7D-21C+49A)x+3F-21D+49B$$

Equating the two sides to get the following equations;

$$C+A=0\cdots(i)$$

$$D+C+B=0\cdots(ii)$$

$$E+D-4C-14A=2\cdots(iii)$$

$$F+E-4D-7C-14B=3\cdots(iv)$$

$$F+3E-7D-21C+49A=1\cdots(v)$$

$$3F-21D+49B=1\cdots(vi)$$

The system consists of six linear equations with six variables. We can use elimination method or otherwise to find the values of A, B, C, D, E, and F.

The values will be

$$A=-\frac{544}{8649},B=-\frac{166}{2883},C=\frac{544}{8649},D=-\frac{46}{8649},E=\frac{128}{93},F=\frac{115}{93}$$

$$\frac{2x^3+3x^2+x+1}{(x^2+x+3)(x^2-7)^2}=-\frac{544x+498}{8649(x^2+x+3)}+\frac{544x-46}{8649(x^2-7)}+\frac{128x+115}{93(x^2-7)^2}$$

$$=\frac{544x}{8649(x^2-7)}-\frac{46}{8649(x^2-7)}+\frac{128x}{93(x^2-7)^2}+\frac{115}{93(x^2-7)^2}-\frac{544x}{8649(x^2+x+3)}-\frac{166}{2883(x^2+x+3)}$$

Remark:

Computer Algebra Systems can be used to decompose rational fractions that are tedious to decompose on paper and pen.

Exercise

1) Decompose into partial fractions;

a) $\frac{x^2+3x+9}{(x-1)(x^2+7)}$

b) $\frac{x^3-1}{(x^2+3)^3}$

c) $\frac{2x^3+2x+7}{(x^2+1)(2x^2+1)^2}$

d) $\frac{x^3+9}{x^2(x^2+x+3)}$

2) Decompose into partial fractions;

a) $\frac{x+2-3x^2}{(x^2+3x-11)^3}$

b) $\frac{7x^3+2x^2+x+1}{(x^2-5)(3x^2-1)}$

c) $\frac{x^2+3x+9}{x^2+3x+17}$

d) $\frac{1-3x-5x^3}{(x^2+1)(x^2-11)^2}$

Bibliography

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