Mathematics for Science

Lecture 9

Trigonometrical Identities and Formulae

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Introduction to Lecture 9

This lecture is a continuation of lecture 8. It will introduce trigonometrical identities and formulae. These concepts are relevant to calculus.

Intended learning outcomes

At the end of this lecture you will be able to;

(a) Identify trigonometrical identities and formulae.

(b) Simplify and apply trigonometrical identities and formulae in solving equations.

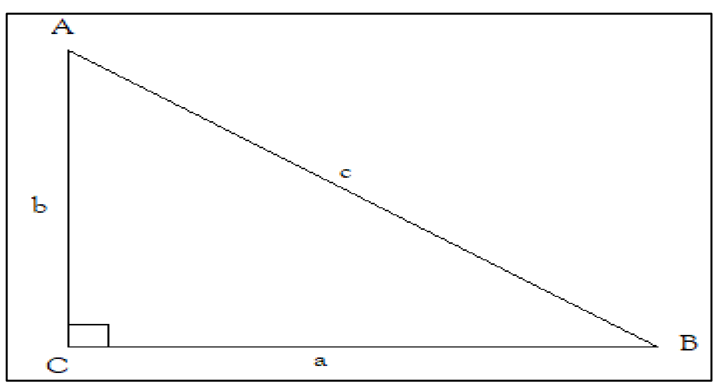
(c) Prove trigonometric identities.

Further readings

The lecture notes can be complemented with relevant topics from Stewart (2012), Swokowski & Cole (2009).

Basic trigonometric identities

Consider the right-angled triangle ABC below;



Applying Pythagoras theorem, we get $c^2 = a^2 + b^2 \cdots \text{(i)}$

Recall from lecture 8 that $\sin B = \frac{b}{c} \Rightarrow b = c \sin B; \cos B = \frac{a}{c} \Rightarrow a = c \cos B$

Substituting a and b in equation (ii) to get;

$$c^2 = (c\cos B)^2 + (c\sin B)^2$$

$$\Rightarrow c^2 = c^2\cos^2 B + c^2\sin^2 B$$

Dividing every term by c^2 to get the trigonometric identity.

$$\cos^2 B + \sin^2 B = 1 \cdots \text{(ii)}$$

Dividing every term in the identity (ii) by $\cos^2 B$ to get;

$$1 + \frac{\sin^2 B}{\cos^2 B} = \frac{1}{\cos^2 B}$$

We simplify the above to get the trigonometric identity.

$$\Rightarrow 1 + \tan^2 B = \sec^2 B \cdots \text{(iii)}$$

Next we divide every term in the identity (i) by $\sin^2 B$ to get;

$$\frac{\cos^2 B}{\sin^2 B} + 1 = \frac{1}{\sin^2 B}$$

We simplify to get the identity;

$$\cot^2 B + 1 = \csc^2 B$$

Example 1: Solve $3\cos^2 x = 3\sin^2 x - \sin x + 1$ where $0^o \leq x \leq 360^o$

Solution: We can rewrite the equation without the cosine. From trigonometric identity $\sin^2 x + \cos^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x$. We replace the cosine in; $3\cos^2 x = 3\sin^2 x - \sin x + 1$ to get;

$$3(1 - \sin^2 x) - 3\sin^2 x + \sin x - 1 = 0$$

$$3 - 3\sin^2 x - 3\sin^2 x + \sin x - 1 = 0$$

$$-6\sin^2 x + \sin x + 2 = 0$$

$$6\sin^2 x - \sin x - 2 = 0$$

Factorizing the above to get; $(2\sin x + 1)(3\sin x - 2) = 0$

$$\Rightarrow 2\sin x + 1 = 0 \text{ or } 3\sin x - 2 = 0$$

$$2\sin x + 1 = 0 \Rightarrow \sin x = -\frac{1}{2} \therefore x = 210^o, 330^o$$

$$3\sin x - 2 = 0 \Rightarrow \sin x = \frac{2}{3} \therefore x \approx 41.8^o, 138.2^o$$

Example 2: Simplify the following expression

$$\frac{1-\cos^2 x}{1-\sin^2 x}$$

Solution:

$$\frac{1-\cos^2 x}{1-\sin^2 x}=\frac{\sin^2 x}{\cos^2 x}=\tan^2 x$$

Example 3: Simplify the following expression

$$\frac{1+2\sin x\cos x}{\sin x+\cos x}$$

Solution:

We can replace 1 with $\sin^2 x+\cos^2 x$ and then factorize the numerator i.e.

$$\frac{1+2\sin x\cos x}{\sin x+\cos x}=\frac{\sin^2 x+\cos^2 x\ +2\sin x\cos x}{\sin x+\cos x}=\frac{(\sin x+\cos x)^2}{(\sin x+\cos x)}$$

$$\Rightarrow\frac{(\sin x+\cos x)(\sin x+\cos x)}{(\sin x+\cos x)}=\sin x+\cos x$$

Example 4: Given that $x=\tan\theta-\cos\theta$ and $y=\tan\theta+\cos\theta$. Show that $x^2-y^2=-4\sin\theta$

Solution: $x^2=(\tan\theta-\cos\theta)^2=\tan^2\theta-2\tan\theta\cos\theta+\cos^2\theta;$

$$y^2=(\tan\theta+\cos\theta)^2=\tan^2\theta+2\tan\theta\cos\theta+\mathrm{Cos}^2\theta$$

Hence $x^2-y^2=(\tan^2\theta-2\tan\theta\cos\theta+\cos^2\theta)-(\tan^2\theta+2\tan\theta\cos\theta+\mathrm{Cos}^2\theta)$

$$=-4\tan\theta\cos\theta$$

$$=-4\sin\theta$$

Example 5: Solve $\tan^2 x-3\sec x+3=0$

Solution: $\sec^2 x-1-3\sec x+3=0$ by replacing $\tan^2 x=\sec^2 x-1$

$$\sec^2 x-3\sec x+2=0$$

Factoring the LHS to get; $(\sec x-1)(\sec x-2)=0$

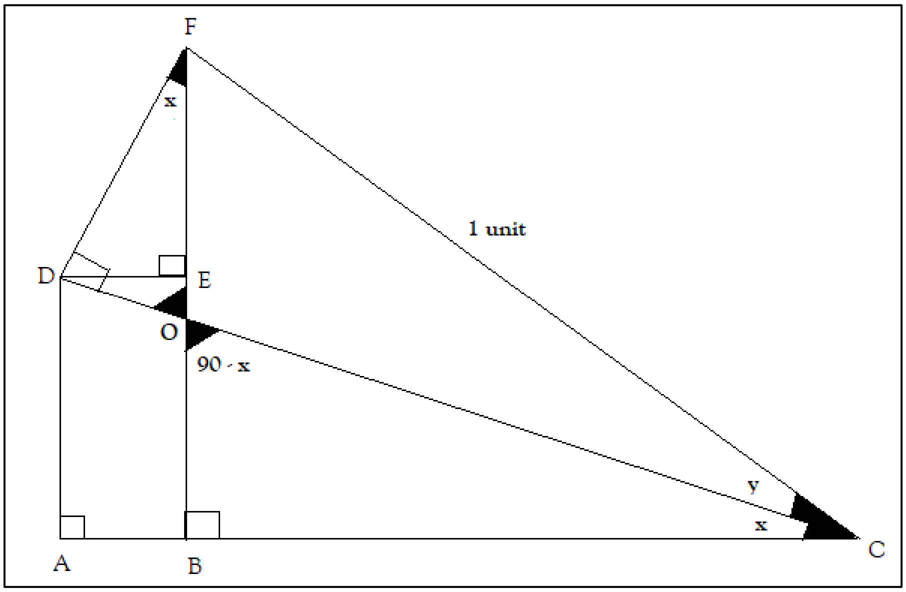
$$\therefore \sec x-1=0\Rightarrow\sec x=1\therefore x=0^o,360^o$$

Also $\sec x-2=0\Rightarrow\sec x=2\therefore x=60^o,300^0$

$$\therefore x=\{0^o,60^o,300^o,360^o\}$$

Compound angle formulae

Consider the figure below;



Triangle CDF is a right-angled triangle with $\angle CDF = 90^o, \angle FCG = y,$ and $\overline{CF} = 1$ unit. Note that triangle CDF has been rotated clockwise about point C through an angle x. $\overline{DA}$ and $\overline{FB}$ are perpendicular to line AC and $\overline{DE}$ is perpendicular to $\overline{FB}$.

$$\sin y = \frac{FD}{1} \Rightarrow FD = \sin y$$

$$\cos y = \frac{CD}{1} \Rightarrow CD = \cos y$$

Note that you can show that $\angle DFE = \angle ACD = x$

Hence considering triangle DFE, $\cos x = \frac{FE}{FD} \Rightarrow FE = FD \cos x = \sin y \cos x$

Next consider triangle ACD, $\sin x = \frac{DA}{CD} \Rightarrow DA = CD \sin x = \cos y \sin x$

Now consider triangle CBF, $\sin(x + y) = \frac{BF}{FC} = \frac{BF}{1} = BF$

$$\Rightarrow \sin(x + y) = BF = FE + EB$$

But EB = DA, therefore

$$\sin(x + y) = FE + DA$$

$$\sin(x + y) = \sin y \cos x + \cos y \sin x$$

Similarly it can be shown that;

(i) $\sin(x - y) = \sin x \cos y - \sin y \cos x$

(ii) $\cos(x + y) = \cos x \cos y - \sin x \sin y$

(iii) $\cos(x - y) = \cos x \cos y + \sin x \sin y$

Since $\tan x = \frac{\sin x}{\cos x}$; it can be shown that;

(iv) $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

(v) $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulae

Suppose $x = y$ then;

$\sin(x + y) = \sin 2x = \sin x \cos x + \sin x \cos x = 2 \sin x \cos x$

$$\Rightarrow \sin 2x = 2 \sin x \cos x \ - \text{Double angle formula}$$

Similarly if we consider $\cos(x + y) = \cos x \cos y - \sin x \sin y$ and let $x = y$ then

$$\cos 2x = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x$$

$$\Rightarrow \cos 2x = \cos^2 x - \sin^2 x$$

$$\tan 2x = \frac{\tan x + \tan x}{1 - \tan x \tan x}$$

$$\Rightarrow \tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Note that in our previous lecture we noted that $\sin^2 x + \cos^2 x = 1$. If we consider the double-angle formula $\cos 2x = \cos^2 x - \sin^2 x$ we can substitute $\cos^2 x$ with $1 - \sin^2 x$ to get;

$$\cos 2x = 1 - \sin^2 x - \sin^2 x$$

$$\Rightarrow \cos 2x = 1 - 2 \sin^2 x$$

Similarly it can be shown that;

$$\cos 2x = 2 \cos^2 x - 1$$

In summary we have the compound and double angle formulae;

a) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

b) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

c) $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

d) $\sin 2x = 2 \sin x \cos x$

e) $\cos 2x = \cos^2 x - \sin^2 x$

f) $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

g) $\cos 2x = 1 - 2 \sin^2 x$

h) $\cos 2x = 2 \cos^2 x - 1$

Example 1: Express $\cos 3x$ in terms of x alone.

Solution: $\cos 3x = \cos(2x + x)$

$$= \cos 2x \cos x - \sin 2x \sin x - \text{by compound angle formula}$$

$$= [2\cos^2 x - 1]\cos x - [2\sin x \cos x]\cos x - \text{by double angle formula}$$

$$= 2\cos^3 x - \cos x - 2\cos^2 x \sin x$$

$$= 2\cos^3 x - 2\cos^2 x \sin x - \cos x$$

$$\Rightarrow \cos 3x = 2\cos^3 x - 2\cos^2 x \sin x - \cos x$$

This is another identity that can be used to simplify working.

Example 2: Express the following in terms of the acute angle; $\cos 135^o$

Solution: We can use the identity in example 1 above i.e.

$$\cos 3x = 2\cos^3 x - 2\cos^2 x \sin x - \cos x$$

$$\cos 135^o = \cos(3 \times 45^o) \Rightarrow x = 45^o$$

Hence we have;

$$\cos 135^o = \cos(3 \cdot 45^o) = 2\cos^3 45^o - 2\cos^2 45^o \sin 45^o - \cos 45^o$$

Recall from previous lecture $\cos 45^o = \frac{1}{\sqrt{2}}$

$$\Rightarrow \cos 135^o = 2\left(\frac{1}{\sqrt{2}}\right)^3 - 2\left(\frac{1}{\sqrt{2}}\right)^2\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

Example 3: Determine the values of $\sin(45^o - 30^o)$

Solution: $\sin(45^o - 30^o) = \sin 45^o \cos 30^o - \cos 45^o \sin 30^o$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

Example 4: Solve the following; $4 \sin x \cos x = 2$ for $0^o \leq x \leq 360^o$

Solution: $4 \sin x \cos x = 2 \Rightarrow 2 \sin x \cos x = 1$

$$\therefore \sin 2x = 1$$

$$\Rightarrow 2x = \sin^{-1}(1) = 90^0, 450^o, \cdots$$

$$\therefore x = 45^o, 225^o$$

Example 5: Find the value of x if $6\tan x = 4\tan(45^o - x)$ for $0^o \leq x \leq 360^o$

Solution: $6\tan x = 4\left(\frac{\tan 45^o - \tan x}{1+\tan 45^o \tan x}\right)$

$$6\tan x = 4\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$6\tan x + 6\tan^2 x = 4 - 4\tan x$$

$$6\tan^2 x + 10\tan x - 4 = 0$$

Factorizing the LHS to get; $(\tan x + 2)(3\tan x - 1) = 0$

$$\Rightarrow \tan x + 2 = 0 \therefore \tan x = -2 \Rightarrow x = \tan^{-1}(-2) \approx -63.4^o \text{ or } 296.6^o$$

$$3\tan x - 1 = 0 \therefore \tan x = \frac{1}{3} \Rightarrow x = \tan^{-1}\left(\frac{1}{3}\right) \approx 18.4^o\ , 198.4^o$$

$$\therefore x = \{18.4^o, 198.4^o, 296.6^o\}$$

Example 6 : Solve $\sin x = \cos 2x$

Solution: $1 - 2\sin^2 x = \sin x \Rightarrow 2\sin^2 x + \sin x - 1 = 0$

Factorizing the LHS to get $(\sin x + 1)(2\sin x - 1) = 0$

$$\Rightarrow \sin x + 1 = 0 \therefore \sin x = -1 \therefore x = 270^o$$

$$\Rightarrow 2\sin x - 1 = 0 \therefore \sin x = \frac{1}{2} \Rightarrow x = 30^o, 150^0$$

Hence $x = \{30^o, 150^o, 270^o\}$

Example 7: Show that $\sin(90^o + x) = \cos x$

Solution: Consider the LHS; $\sin(90^o + x) = \sin 90^o \cos x + \cos 90^o \sin x$

$$= \cos x + 0 \cdot \sin x = \cos x$$

Example 8: Prove that $2\cos x = \frac{1+\sin 2x+\cos 2x}{\cos x+\sin x}$

Solution: Consider the RHS. We simplify it and show it is equal to the LHS

$$\frac{1+\sin 2x+\cos 2x}{\cos x+\sin x} = \frac{1+2\sin x\cos x+2\cos^2 x-1}{\cos x+\sin x} = \frac{2\sin x\cos x+2\cos^2 x}{\cos x+\sin x}$$

$$= \frac{2\cos x(\sin x+\cos x)}{(\cos x+\sin x)} = 2\cos x$$

Exercise

1) Show that $\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$ i. e. the factor formula.

2) Show that $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ i. e. halfangle formula.

3) Solve $3\tan\theta - \cot\theta = 2$ for $0^o \leq \theta \leq 360^o$

4) Evaluate with using a scientific calculate ; $2\sin 105^o \cos 105^o$

5) Show that $(\cos\theta + \sin\theta)^2 - \sin 2\theta = 1$

6) Solve $\tan(x + 20^o) = \sin 40.5^o$ for $0^o \leq x \leq 360^o$

7) Solve $3\cos A + 4\sin A = 5$ for $0^o \leq A \leq 360^o$

8) Given that $\cos A = \frac{12}{13}$ and $\cos B = 0.6$ find the values of;

$\cos(A + B)$; $\sin(A + B)$, $\cos(A - B)$, $\sin(A - B)$, and $\tan(A + B)$ (*Take A and B as acute angles).*

Bibliography

Stewart, J. (2012). *Calculus* (7th ed.). BROOKS/COLE Cengage Learning.

Swokowski, E., & Cole, J. A. (2009). *Algebra and Trigonometry with Analytical Geometry*. BROOKS/COLE Cengage Learning.