Mathematics for Science

Lecture 10

Introduction to Permutations

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Introduction to Lecture 10

The lecture will introduce the concept of permutations. These concepts are relevant to understanding concepts in discrete mathematics which is fundamental in computer science and/or technology related courses.

Intended learning outcomes

At the end of this lecture, you will be able to;

(a) Define a permutation.

(b) Apply the concept of permutation.

Further readings

The lecture notes can be complemented with relevant topics from (Kahenya, 2017; Mott et al., 2004; Murray & Robert, 2009).

Introduction

Consider three towns A, B, and C that are connected by different roads. Suppose there are 4 roads from town A to town B and 7 roads from town B to town C. One can use 4×7 different roads from town A to town C through town B (Kahenya, 2017).

In general if event e_1 can be done in x_1 ways, event e_2 in x_2 ways, event e_3 in x_3 different ways, and so on.

Then event $e_1 e_2 e_3 \cdots e_n$ can be done in $x_1 \times x_2 \times x_3 \times \cdots \times x_n$ ways.

Definition 1: A permutation is an ordered arrangement of all or part of a set objects.

Definition 2: A permutation of n objects taken r at a time is an ordered selection or arrangement of r objects (Mott et al., 2004).

Example 1: How many different ways can the letters a,b,c be arranged without repeating any letter.

Solution: The letters can be arranged to form the following;

$$\text{abc, acb, bac, bca, cab, cba}$$

these arrangements are called permutations.

Definition 3: In general, n factorial denoted $n!$ Is the product of the first n natural numbers that is; $n! = n(n-1)(n-2)(n-3)\cdots 4 \times 3 \times 2 \times 1$.

Note that zero factorial is equal to 1 i.e. $0! = 1$. Factorial exists only for whole numbers.

Example 1: Evaluate $7!$

Solution: $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

Example 2: Consider the eight boys who enter a room with only 3 seats placed in a row. Determine in how many ways can the seats be occupied.

Solution: our interest here is how the three seats can be occupied and NOT how the 8 boys can be seated. Therefore the 1^{st} seat can be occupied by any of the 8 boys i.e. 8 different ways. After the first seat is occupied the 2^{nd} seat can be occupied in 7 different ways (7 remaining boys), and the 3^{rd} seats can be occupied in 6 different ways. Hence in total, the 3 seats can be occupied in $8 \times 7 \times 6 = 336$ different ways.

Example 3: Suppose we have 8 seats in a row and 3 boys. Determine how many ways can the 3 boys be seated.

Solution: Our interest now is the boy and NOT the seats. The 1^{st} boy has 8 different ways/seats to be arranged. After he takes a seat then 2^{nd} boy will have 7 different seats/ways to be arranged, and the 3^{rd} boy will have 6 different seats/ways to be arranged.

In total the 3 boys will have $8 \times 7 \times 6 = 336$ ways

In the two examples, even though the solution is the same, the approach is different.

Definition 4: In general, to determine the number of unusual ways or arrangements or permutations of n different objects taken r at a time given by;

$$P(n,r) = {}_nP_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Example 1: How many different ways can 11 boys occupy 5 seats in a row?

Solution: ${}_nP_r = {}_{11}P_5 = \frac{11!}{(11-5)!} = \frac{11!}{6!} = 55440$ ways

Example 2: Find the value of n if ${}_nP_4 = {}_nP_3$

Solution:

$${}_nP_4 = \frac{n!}{(n-4)!} = \frac{n(n-1)(n-2)(n-3)(n-4)!}{(n-4)!} = n(n-1)(n-2)(n-3)$$

$${}_nP_3 = \frac{n(n-1)(n-2)(n-3)!}{(n-3)!} = n(n-1)(n-2)$$

$$\Rightarrow n(n-1)(n-2)(n-3) = n(n-1)(n-2)$$

$$n-3 = 1 \therefore n = 4$$

Example 3: Four different editions of National Geographic Magazines and six other different magazines are to be arranged on a magazine rack. Determine;

(i) the number of possible arrangements of the magazines.

(i) The number of possible arrangements if the National Geographic magazines are always together.

Solution:

(i) The 10 magazines can be arranged in $10!$ Ways i.e. 3628800 ways

(ii) We shall treat the National Geographic magazines as 'one magazine' since they must always be together. Hence we shall have '7 magazines' in total that can be arranged in 7! Ways i.e. 5040 ways.

However the 4 National Geographic magazines can be arranged 4! Ways i.e. 24 ways. Therefore the total arrangements are;

$$4! \times 7! = 5040 \times 24 = 120960 \text{ ways}$$

Definition 5: Permutations p of n objects all in a row taken together when n_1 are alike of first kind, n_2 are alike of second kind, n_3 are alike of third kind, and so on is given by;

$$p = \frac{n!}{n_1!\,n_2!\,n_3!\cdots} \text{ where } n_1 + n_2 + n_3 + \cdots = n$$

Example 1: Determine the number of ways 5 mangoes and 7 oranges can be distributed among 12 boys to receive a fruit.

Solution: Our number of objects $n = 12, n_1 = 5,$ and $n_2 = 7$

Hence the permutations $p = \frac{12!}{5!7!} = \frac{12\times11\times10\times9\times8\times7!}{5\times4\times3\times2\times1\times7!} = \frac{12\times11\times10\times9\times8}{5\times4\times3\times2\times1} = 792$

Example 2: In how many ways can 5 red marbles, 3 green marbles, and 7 black marbles can be arranged in a row (where marbles of the same color are indistinguishable).

Solution: The total number of marbles are 15. Hence the number of arrangements

$$= \frac{15!}{5!\,3!\,7!} = 360360 \text{ ways}$$

Definition 6: (Circular permutations)

The number of different ways of arranging n objects around a circle (e.g. a round table etc.) is given by;

$$(n-1)!$$

The idea is that 1 object is fixed and the rest are arranged relative to it.

Example 1: Find the number of ways 7 boys can be seated around a circular table.

Solution: Our $n = 7$ therefore we have; $(n-1)! = (7-1)! = 6! = 720$ different ways

Definition 7: The number of different ways of arranging n objects around a ring is given by;

$$\frac{(n-1)!}{2}$$

It is divided by 2 since he clockwise and anticlockwise arrangement are not considered to be different (e.g. in a ring this same permutation can be achieved by flipping over the ring).

Example 1: Find the different ways of arranging 20 beads in a necklace.

Solution: The number of different ways is;

$$\frac{(n-1)!}{2} = \frac{19!}{2}$$

Definition 7: (Mutually Exclusive scenarios)

When two events X and Y are mutually exclusive then, if event X occurs then Y cannot occur, and if Y occurs, X cannot occur. Hence the number of permutations of either X or Y occurring is the sum of the number of permutations of X and the number of permutations of Y.

Example 1: How many different 4-letter codes can be formed from the letters a, b, c, d with no repetition and the code;

(i) Ends with b

(ii) Ends with d

(iii) Ends with b or d

Solution:

(i) The 4^{th} letter can be chosen in 1 way only since it must be b. The 1^{st} letter 3 ways, the 2^{nd} 2 ways and the 3^{rd} in 1 way. Hence we have; $3 \times 2 \times 1 \times 1 = 6$ different ways

(ii) The 4^{th} letter can be chosen in 1 way only since it must be d. The 1^{st} letter 3 ways, the 2^{nd} 2 ways and the 3^{rd} in 1 way. Hence we have; $3 \times 2 \times 1 \times 1 = 6$ different ways

(iii) The code that ends with a b cannot also end with d, hence the codes are mutually exclusive. Therefore we have $6 + 6 = 12$ different ways.

Note that (iii) can also be argued this way: the 4^{th} letter can be arranged in 2 different ways. The 1^{st} letter in 3 ways, the 2^{nd} letter in 2 ways and the 3^{rd} letter in 1 way. Hence we get;

$3 \times 2 \times 1 \times 2 = 12$ different ways.

Example 2: The Chair, the secretary, and three members of a certain committee are seated around a circular table. Determine in how many different ways they can be arranged if;

(i) One can sit next to anyone else.

(ii) The secretary must sit next to the Chair.

(iii) The secretary must not sit next to the Chair.

Solution:

(i) The sitting positions does not matter and therefore we have;

$$(n-1)! = (5-1)! = 4! = 24 - \text{different ways}$$

(ii) Since the secretary and the Chair MUST sit next to each other, consider this as '1 person'. Hence you need to arrange 4 people around a circular table i.e.

$$(n-1)! = (4-1)! = 3! = 6 - \text{different ways}$$

However the secretary and the Chair have2 different ways of sitting. Therefore the total number of different ways of sitting the committee is;

$$2 \times 6 = 12 - \text{different ways}$$

(iii) The scenario where the secretary does not sit next to the Chair is a mutually exclusive one. Hence the of permutations will be the same as the total number of permutations (where there is no uniqueness in the sitting positions) less the number of permutations when the secretary sit next to the Chair i.e.

$$24 - 12 = 12 - \text{different ways}$$

Exercise

1) Evaluate; $19!$; $12!$; $\frac{8!}{9!}$; $\frac{32!}{33!}$
2) Evaluate $p(n, r)$;
   (i) $p(12,5)$
   (ii) $p(17, 11)$
   (iii) $p(9,2)$
3) Show that $p(n, n) = n!$
4) In how many different ways can a group of 11 boys be arranged in a row?
5) Determine how many 4-digit codes can be formed from the numbers 1,2,3,and 4 if;
   (i) Each number is used only once in each code.
   (ii) If repetition of number is allowed.
6) Four boys and 3 girls are seated on a bench. Two of them are brother and sister. Determine in how many ways can they be seated if the siblings must sit together.
7) Twenty boys are racing. In how many different ways can the first 3 positions be won?
8) Find how many ways can 4 red pencils, 3 yellow pencils, and 2 black pencils be arranged in a row if the pencils of the same colour are indistinguishable.
9) The letters of the word AVOCADO are arranged in a row. Find;
   (i) How many different ways can the letters be arranged.
   (ii) How many ways can the letters be arranged if the two A's are together.
   (iii) How many ways can the letters be arranged if the two A's and the two O's must be together.

Bibliography

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