Mathematics for Science

Lecture 11

Permutation Functions

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Introduction to Lecture 11

This lecture is a continuation of Lecture 10. The lecture will introduce the concept of permutations as functions. These concepts are relevant to understanding concepts in discrete mathematics and modern algebra (abstract algebra especially group and ring theory). Actually the permutation functions form the symmetric groups.

Intended learning outcomes

At the end of this lecture, you will be able to;

(a) Define a permutation function.

(b) Carry out operations involving permutations functions.

Further readings

The lecture notes can be complemented with relevant topic from (Judson & Beezer, 2018; Pinter, 2010; William & Nicholson, 2004).

Definition 1: A bijection or a one-to-one and onto mapping, from a set X to itself is called a permutation of X.

Definition 2: If set $X = \{x_1, x_2, x_3, \cdots, x_n\}$ is a finite set and p is a bijection on X, we list the elements of X and the corresponding function values $p(x_1), p(x_2), \cdots p(x_n)$ in the following form;

$$\begin{pmatrix} x_1 & x_2 \cdots & x_n \\ p(x_1) & p(x_2) \cdots & p(x_n) \end{pmatrix}$$

Note that the above describes p since it gives the value of p for every element of X. we denote it as;

$$p = \begin{pmatrix} x_1 & x_2 \cdots & x_n \\ p(x_1) & p(x_2) \cdots & p(x_n) \end{pmatrix}$$

In short, if p is a permutation of a finite set $X = \{x_1, x_2, \cdots, x_n\}$ then the sequence $p(x_1), p(x_2), \cdots p(x_n)$ is simply a rearrangement of the elements of X and hence corresponds exactly to a permutation of X.

Example 1: Let $A = \{a, b, c\}$ then all the permutations of A are;

$I_n = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}$, $p_1 = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}$, $p_2 = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$, $p_3 = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$, $p_4 = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}$,

$p_5 = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$

Compare with the arrangements abc, acb, bac, bca, cab, and cba.

Definition (Inverse permutation): If p is a permutation defined on a finite set say A with n elements, then by definition p is a one-one mapping of A onto itself. Since p is a one-one onto mapping then it is invertible. Its inverse p^{-1} is also a one-one mapping of A onto itself. Therefore if;

$$p = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & \cdots & b_n \end{pmatrix}$$

then

$$p^{-1} = \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & \cdots & b_n \\ a_1 & a_2 & a_3 & \cdots & \cdots & a_n \end{pmatrix}$$

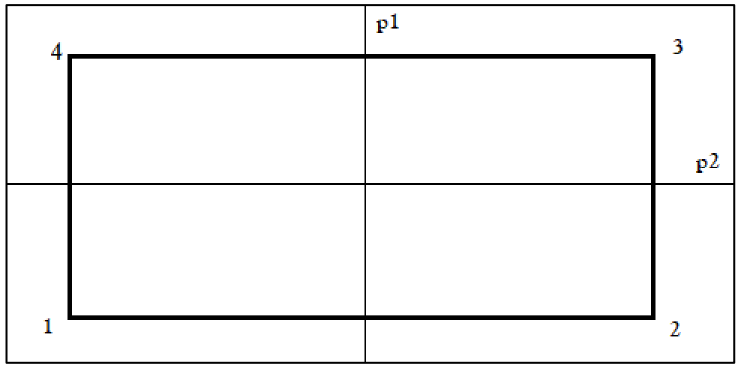
Remark: It is clear that p^{-1} is obtained by interchanging the rows of p. Since $p(a_1) = b_1 \Rightarrow p^{-1}(b_1) = a_1$

Example 2: Find p_3^{-1}

Solution: Let us view $p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ in the form $p_3 = \{(1,2), (2,3), (3,1)\}$ therefore the inverse of p_3 i. e. $p_3^{-1} = \{(2,1), (3,2), (1,3)\}$. Note that p_3^{-1} as a permutation function is

$$p_3^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Example 3: Consider the rectangle with vertices 1,2,3,and 4 as shown below;



Let p1 - reflection along the line of symmetry p1

Let p2 - reflection along the line of symmetry p2

Let α_1 - rotation of $+180^o$ about the centre.

Let α_0 – rotation of $+360^o$ or 0^o about the centre.

Then we have the permutations;

$\alpha_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ $\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$, $p1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$, $p2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

Find $p1 \circ \alpha_1$

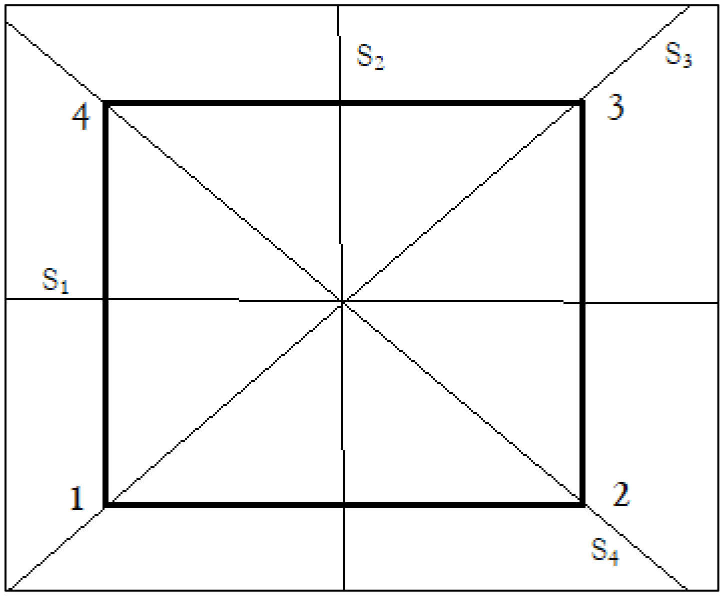
Solution:

$p1 \circ \alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$;

$\alpha_1 \circ p1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

Note that $p1 \circ \alpha_1 = \alpha_1 \circ p1$ even though this is not always the case.

Example 4: Consider the square with vertices 1,2,3, and 4 as shown below;



Suppose we rotate the square about its centre and reflect it along the lines of symmetry $S_1, S_2, S_3,$ and S_4, we get the following permutations;

<u>Rotations</u>

Let r_0 be a rotation of 360^o or 0^o about the centre $r_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

Let r_1 be anticlockwise rotation of 90^o about the centre $r_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

Let r_2 be anticlockwise rotation of 180^o about the centre $r_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

Let r_3 be anticlockwise rotation of 270^o about the centre $r_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

<u>Symmetry</u>

Let S_1 be reflection on the line of symmetry $S_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

Let S_2 be reflection on the line of symmetry $S_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

Let S_3 be reflection on the line of symmetry $S_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$

Let S_4 be reflection on the line of symmetry $S_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

Example 5: Consider the square in the previous example. Determine the permutation for reflecting the square along the line of symmetry S_3 then followed by anticlockwise rotation of

270^{o} about the centre and then an inverse anticlockwise rotation of 90^{o} about the centre of the square.

Solution: This is the composite function $r_1^{-1} \circ r_3 \circ S_3$. Note that the order is important.

$$r_1^{-1} \circ r_3 \circ S_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} = S_4$$

This is the same as reflecting the square along the axis S_4

Definition 4: (Cyclic permutation) Let $x_1, x_2, \cdots, x_r$ be r distinct elements of the set $X = \{x_1, x_2, x_3, \cdots, x_n\}$. The permutation $p: X \to X$ defined by;

$$p(x_1) = x_2$$
$$p(x_2) = x_3$$
$$p(x_3) = x_4$$
$$\cdots$$
$$p(x_{r-1}) = x_r$$
$$p(x_r) = x_1$$
$$p(x) = x \text{ if } x \in X, x \notin \{x_1, x_2, \cdots, x_r\}$$

is called a cyclic permutation of length r or simply a cycle of length r denoted by $(x_1 x_2 x_3 \cdots x_r)$

Example 1: Let $A = \{1,2,3,4,5,6\}$ then the cycle (3521) denotes the permutation;

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 4 & 2 & 6 \end{pmatrix}$$

(i) The elements in a cycle cannot tell the number of elements in the set. The set need to be explicitly defined.

(ii) The fixed elements are the elements missing in the cycle. For example in the cycle above elements 4 and 6 are fixed.

(iii) The product of two cycles need not be a cycle.

Example 2: Let $A = \{1,2,3,4,5,6\}$ compute $(145) \circ (5136)$

Solution: $(145) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 3 & 5 & 1 & 6 \end{pmatrix}$, $(5136) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 6 & 4 & 1 & 5 \end{pmatrix}$

$$\Rightarrow (145) \circ (5136) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 3 & 5 & 1 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 6 & 4 & 1 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 6 & 5 & 4 & 1 \end{pmatrix} = (136)(45)$$

Definition 5: (Disjoint cycles): Two cycles of a set A are said to be disjoint if they have no common element of A.

Example 1: Let $A = \{1,2,3,4,5,6\}$ then the cycles (24) and (35) are disjoint whereas cycles (236) and (146) are not.

Theorem 1: A permutation of a finite set that is not the identity or a cycle can be written as a product of disjoint cycles of length ≥ 2

Example 1: Let $A = \{1,2,3,4,5,6,7\}$ and its permutation $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 6 & 7 & 4 & 5 \end{pmatrix}$. We can write p as a product of disjoint cycles i.e. $p = (123) \circ (46) \circ (57)$

Note:

a) If $p_1 = (a_1 a_2 \cdots a_r)$ and $p_2 = (b_1 b_2 \cdots b_s)$ are disjoint cycles of set A then $p_1 \circ p_2 = p_2 \circ p_1$

b) When a permutation is written as a product of disjoint cycles, the product is unique except for the order of the cycles.

Example 2: Given set $X = \{1,2,3,4,5\}$ compute $(15) \circ (234)$ and $(234) \circ (15)$

Solution:

$$(15) \circ (234) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 2 & 1 \end{pmatrix}$$

$$(234) \circ (15) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 2 & 1 \end{pmatrix}$$

It is clear that $(15) \circ (234) = (234) \circ (15)$

Definition 6: (Transposition)

A cycle of length 2 is called a transposition. A transposition is a cycle $p = (x_i x_j)$ where $p(x_i) = x_j$ and $p(x_j) = x_i$.

Corollary 1: Every permutation of a finite set with at least two elements can be written as a product of transpositions. Note that the transpositions need not be disjoint.

Example 1: Let $p = (57) \circ (123) \circ (235)$

Note that $(123) = (12) \circ (13)$ and $(235) = (25) \circ (23)$.

Then $p = (57) \circ (12) \circ (13) \circ (25) \circ (23)$

Definition 7: (Even Permutation)

A permutation of a finite set is called even if it can be written as a product of an even number of transpositions.

Example 1: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix} = (235) = (25)(23)$

Definition 8: (Odd Permutation)

A permutation of a finite set is called odd if it can be written as a product of an odd number of permutations.

Example 1: Consider the permutation $\mathrm{p} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 5 & 7 & 6 & 3 & 1 \end{pmatrix}$ is p odd or even?

Solution: p is a product of disjoint cycles; $\mathrm{p} = (1247)(356)$

If we rewrite each cycle as a product of transpositions we have;

$(1247) = (17) \circ (14) \circ (12)$ and $(356) = (36) \circ (35)$

$\Rightarrow \mathrm{p} = (36) \circ (35) \circ (17) \circ (14) \circ (12) \Rightarrow \mathrm{p}$ is an odd permutation.

Definition 9: A permutation group of a set A is a set of permutations of A that forms a group under function composition.

Definition 10: The symmetric group $\mathrm{S_n}$ is the group of all permutations of the set $\{1,2,3, \cdots n\}$

Example 1: The permutations of rotating and reflecting an equilateral triangle about its centre and along its line of symmetries respectively form the symmetric group S_3. Note that within the symmetric groups we may have subgroups.

Definition 11: The subset of even permutations in the symmetric group $\mathrm{S_n}$ is a subgroup called the alternating group $\mathrm{A_n}$.

Example 1: The group $\mathrm{A_4}$ is the subgroup of $\mathrm{S_4}$ consisting of even permutations. There are twelve elements in $\mathrm{A_4}$

I	(12)(34)	(13)(24)	(14)(23)
(123)	(132)	(124)	(142)
(134)	(143)	(234)	(243)

Exercise

1) Given $p_1 = \begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 4 & 3 & 2\end{pmatrix}$, $p_2 = \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}$, $p_3 = \begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 1 & 2\end{pmatrix}$, $p_4 = \begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 4 & 2\end{pmatrix}$, $p_5 = \begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 2 & 4 & 1\end{pmatrix}$ work out;

a) p_4^{-1}
b) p_2^{-1}
c) p_3^{-1}
d) $p_4 \circ p_3 \circ p_1$
e) $p_3 \circ p_2 \circ p_4^{-1}$
f) $p_3 \circ p_5 \circ p_1$

2) Write the following permutations in cycle notations

a) $\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 1 & 2 & 4 & 5 & 3\end{pmatrix}$
b) $\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 4 & 5 & 3 & 1 & 2\end{pmatrix}$
c) $\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 5 & 1 & 4 & 3 & 2\end{pmatrix}$
d) $\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 6 & 3 & 2 & 5 & 4 & 1\end{pmatrix}$

3) Compute each of the following;

a) $(125)(254)$
b) $(142)(123)$
c) $(153)(53)(241)$
d) $(3254)(43)(15)$

4) Express the following as product of transpositions

a) (234)(125)(367)
b) (235)(189)(256)(2675)

5) Let set $A = \{1,2,3,4,5,6\}$ and permutation $p = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 2 & 1 & 4 & 3 & 6 & 5\end{pmatrix}$ be a permutation of set A.

a) Write p as a product of disjoint cycles.
b) Compute p^{-1}
c) Compute p^2
d) Is permutation p odd or even?

6) Work out; $\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 4 & 5 & 2 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 4 & 3 & 5 & 1 & 2\end{pmatrix}^{-1}$

7) List all the permutation functions that are generated by rotating an equilateral triangle and a regular hexagon about their centres and reflection along their lines of symmetry.

Bibliography

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