Mathematics for Science

Lecture 12

Introduction to Combinations

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Introduction to Lecture 12

The lecture will introduce the concept of combinations. It is a continuation of lecture 10 on permutations. These concepts are relevant to understanding concepts in discrete mathematics which is fundamental in computer science and/or technology related courses.

Intended learning outcomes

At the end of this lecture, you will be able to;

(a) Define a combination.
(b) Differentiate between permutations and combinations.
(c) Apply the concept of combinations.

Further readings

The lecture notes can be complemented with relevant topics from (Kahenya, 2017; Mott et al., 2004; Murray & Robert, 2009).

Definition 1: A combination is a selection of all or part of objects without regard to order.

Remark 1: In lecture 10 we noted that a permutation is an ordered arrangement of objects. On the other hand, a combination is a selection of objects with no regard to order.

Example 1: A combinations of three numbers 1,2, and 3 taking 2 at a time are 12, 13, and 23. The option 21 and 12 is one combination but two permutations of 1 and 2.

Example 2: How many selections of 3 letters can be made from 6 letters a, b, c, d, e, f ?

Solution: Suppose there are n selections or combinations. Each selection of 3 letters can be arranged in $3!$ ways e.g. the selection containing a, b, c can be arranged to give;

$$abc, acb, bac, bca, cab, cba$$

$\Rightarrow$ for n selections there are $3!n$ arrangements , but the total number of arrangements of the 6 letters taken three at a time is;

$$^6P_3$$

$$\Rightarrow 3!n = {}^6P_3 = {}^6C_3$$

$$n = \frac{^6P_3}{3!}$$

$$n = \frac{6\times5\times4}{3\times2\times1} = 20$$

Definition 2: A combinations of n different objects taken r at a time is given by;

$$_n\mathrm{C}_r = \frac{_n\mathrm{P}_r}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!\,(n-r)!}$$

Example 1: How many different teams of 7 students can be chosen from a team of 20 students.

Solution: The different teams is the number of combinations of 7 students out of a population of 20 students i.e.

$$_{20}\mathrm{C}_7 = \frac{20!}{7!\,(20-7)!} = \frac{20!}{7!\,13!} = \frac{20\cdot19\cdot18\cdot17\cdot16\cdot15\cdot14\cdot13!}{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1\cdot13!}$$

$$= \frac{20\cdot19\cdot18\cdot17\cdot16\cdot15\cdot14}{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}$$

$= 77520$ selections

Example 2: In how many ways can a wall designer select 5 paintings out of 12 paintings?

Solution: $_{12}\mathrm{C}_5 = \frac{12!}{5!(12-5)!} = \frac{12!}{5!7!} = \frac{12\cdot11\cdot10\cdot9\cdot8\cdot7!}{5\cdot4\cdot3\cdot2\cdot1\cdot7!} = \frac{12\cdot11\cdot10\cdot9\cdot8}{5\cdot4\cdot3\cdot2\cdot1} = 792$ ways

Example 3: in how many ways can one select 3 pencils out of 7 pencils.

Solution: $_7\mathrm{C}_3 = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = \frac{7\times6\times5\times4!}{3!4!} = 35$ ways

Definition 3: The number of selections r out of n objects is equal to the number of selections of $(n - r)$ of on n objects. That is, ${}_{n}C_{r} = {}_{n}C_{(n-r)}$

Example 1: ${}_{12}C_{5} = {}_{12}C_{7}$

${}_{12}C_{7} = \frac{12!}{7!5!} = \frac{12\cdot11\cdot10\cdot9\cdot8\cdot7!}{5\cdot4\cdot3\cdot2\cdot1\cdot7!} = \frac{12\cdot11\cdot10\cdot9\cdot8}{5\cdot4\cdot3\cdot2\cdot1} = 792$ ways

Example 3: How many different teams each consisting of 5 boys and 2 girls can be chosen from a group of 11 boys and 6 girls?

Solution: The number of ways of choosing 5 boys from 11 boys is given by;

${}_{11}C_{5} = \frac{11!}{(11-5)!5!} = 462$

The number of ways of choosing 2 girls from 6 girls is given by;

$${}_{6}C_{2} = \frac{6!}{(6-2)!\,2!} = 15$$

Hence the total number of teams;

$$462 \times 15 = 6930$$

Example 3: A committee of 10 members is to be formed from a group of 10 parents, 7 faculty, and the VC. In how many ways can the committee be formed in order to include;

(i) The VC

If the first slot is taken by the VC we shall remain with 9 slots. Hence

$${}_{17}C_{9} = \frac{17!}{(17-9)!\,9!} = 24310$$

(ii) The VC and 5 parents

Numbers of ways of selecting 5 parents from a pool of 10 parents is given by; ${}_{10}C_{5} = 252$

Number of ways of selecting 4 faculty (since the 6 slots have already been taken by the VC and 5 parents) from a pool of 7 faculty is given by; ${}_{7}C_{4}$

Therefore, the number of ways that a committee can be formed to include the VC and 5 parents is given by;

$$1 \times {}_{10}C_{5} \times {}_{7}C_{4} = 1 \times \frac{10!}{(10-5)!\,5!} \times \frac{7!}{(7-4)!\,4!} = 1 \times 252 \times 35 = 8820$$

Definition 4: The total number of combinations C of n different items taken $1,2,3,\cdots n$ at a time is;

$$C = 2^n - 1$$

Example 1: In how many ways can a boy choose to buy 1 or more of the phones: Samsung, Iphone, Huwei, Techno, and Infinix.

Solution: The boy has two ways to consider for each phone i.e. each phone can be chosen or not. Therefore, each of the 2 ways of dealing with a phone is associated with 2 ways of dealing with the others. Hence, we shall have the total number of ways to be;

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5$$

However, this includes the scenario where no phone is chosen.

The numbers of ways become: $2^5 - 1 = 31$

Alternative approach:

The boy can opt to buy 1 phone or 2 or 3 or 4 or 5. That is;

$$_5C_1 + \ _5C_2 + \ _5C_3 + \ _5C_4 + \ _5C_5$$

$$= \frac{5!}{(5-1)!\,1!} + \frac{5!}{(5-2)!\,2!} + \frac{5!}{(5-3)!\,3!} + \frac{5!}{(5-4)!\,4!} + \frac{5!}{(5-5)!\,5!}$$

$$= \frac{5!}{4!} + \frac{5!}{3!\,2!} + \frac{5!}{2!\,3!} + \frac{5!}{1!\,4!} + \frac{5!}{0!\,5!} = 5 + 10 + 10 + 5 + 1 = 31$$

Example 2: The integers 1,2, 3,...9 are to be divided into 3 groups so as to create codes containing 2, 3, and 4 integers respectively. Find the different ways that this can be achieved.

Solution: The two integers for the first set of codes can be chosen; $_9C_2$

The next set of 3 integers can be chosen; $_7C_3$

The remaining integers can only be chosen once to form a 4-integers code.

Hence the total number of different ways are

$$_9C_2 \times \ _7C_3 \times 1 = \frac{9!}{7!\,2!} \times \frac{7!}{4!\,3!} \times 1 = 1260$$

Remark: The different ways of dividing $p_1 + p_2 + p_3 + \cdots + p_n$ unlike objects into n sets of objects that have $p_1, p_2, \cdots p_n$ objects respectively is given by;

$$\frac{(p_1 + p_2 + p_3 + \cdots + p_n)!}{p_1!\,p_2!\,p_3! \cdots p_n!}$$

Note that the size of sets may vary.

Example 3: Find the number of ways that one can arrange 24 crayons into sets if there are to be:

(i) 2 sets of 12 crayons.

(ii) 3 sets of 8 crayons.

Solution:

(i) The first set of 12 crayons can be chosen in $_{24}C_{12}$

The next set of 12 crayons can only be chosen once.

However, the crayons in the first set can also be in the second set. Hence, we need to divide the numbers of ways in which the 2 sets can be permutated amongst themselves i.e. 2!

Therefore, the number of different ways of grouping the crayons into 2 sets of crayons is given by;

$$\frac{_{24}C_{12} \times 1}{2!} = 1352078$$

(ii) The first set of 8 crayons can be selected in $_{24}C_8$

The second set of 8 crayons will be selected in $_{16}C_8$

The third set of 8 crayons can only be selected in one way.

The 3 sets can be permutated amongst the crayons in 3! ways.

Therefore, the number of different ways of grouping the crayons into 3 sets is given by;

$$\frac{_{24}C_8 \times \ _{16}C_8 \times 1}{3!} = 1577585295$$

Example 4: Solve the following $_nC_5 = \ _nC_4$

Solution:

$$_nC_5 = \frac{n!}{(n-5)!\,5!} = \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)!}{(n-5)!\,5!} = \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}$$

$$_nC_4 = \frac{n!}{(n-4)!\,4!} = \frac{n(n-1)(n-2)(n-3)(n-4)!}{(n-4)!\,4!} = \frac{n(n-1)(n-2)(n-3)}{4!}$$

$\Rightarrow \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} = \frac{n(n-1)(n-2)(n-3)}{4!}$

$\Rightarrow n(n-1)(n-2)(n-3)(n-4)4! = n(n-1)(n-2)(n-3)5!$

$= (n-4)4! = 5!$

$24n - 96 = 120 \therefore n = 9$

Example 5 : Find the value of n if $\,_{n}C_4 = \,_{n}C_3$

Solution:

$$\,_{n}C_4 = \frac{n!}{(n-4)!\,4!} = \frac{n(n-1)(n-2)(n-3)(n-4)!}{(n-4)!\,4!} = \frac{n(n-1)(n-2)(n-3)}{4!}$$

$$\,_{n}C_3 = \frac{n(n-1)(n-2)(n-3)!}{(n-3)!\,3!} = \frac{n(n-1)(n-2)}{3!}$$

$$\Rightarrow \frac{n(n-1)(n-2)(n-3)}{4!} = \frac{n(n-1)(n-2)}{3!}$$

$$n(n-1)(n-2)(n-3)3! = n(n-1)(n-2)4!$$

$$6(n-3) = 24 \Rightarrow \; n-3 = 4 \therefore n = 7$$

Exercise

1) Evaluate the following.

   (i) ${}_{10}C_8$

   (ii) ${}_{8}C_3$

   (iii) $\frac{{}_{13}C_5}{{}_{19}C_7}$

2) A team consists of 7 boys and 9 girls. Determine in how many different ways a team of five can be selected such that it contains.

   (i) No boys

   (ii) At least a boy

   (iii) 2 boys and 3 girls

   (iv) 3 boys and 2 girls

   (v) No girls

   (vi) At least a girl

3) Find the number of different selections of 3 letters can be made from the letters of the word MISSISSIPPI.

4) A group of 18 students is to be divided into 3 teams of 5, 6, and 7 students. In how many ways can this be achieved?

5) Find in how many ways can 15 children be divided into;

   (i) A team of 5 and a team of 10.

   (ii) Three teams of 5 members each.

6) Solve the for n;

   (i) ${}_{n}C_5 = {}_{n}C_4$

   (ii) ${}_{n}C_{12} = {}_{n}C_8$

Bibliography

Kahenya, P. (2017). *Foundation Maths*. LAP Lambert Academic Publishers.

Mott, J. L., Kandel, A., & Baker, T. P. (2004). *Discrete Mathematics for Computer Scientists & Mathematicians* (second). Prentice Hall of India.

Murray, S., & Robert, M. (2009). *College Algebra*. McGraw-Hill.