Mathematics for Science

Lecture 13

Introduction to Mathematical Induction

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Introduction to Lecture 13

The lecture will introduce one of the methods of indirect proof i.e. proof by mathematical induction. The principle of mathematical induction is used to establish the truth of propositions or statements that are defined over the set of positive integers or natural numbers.

Intended learning outcomes

At the end of this lecture, you will be able to;

(a) Define the principle of mathematical induction.

(b) Apply the principle of mathematical induction to prove theorems.

Further readings

The lecture notes can be complemented with relevant topics from (Kahenya, 2017; Mott et al., 2004; Murray & Robert, 2009; Stewart, 2012)

Definition 1: A proposition is a statement that can be assigned a truth value e.g. today is Friday, A cow has a tail, $3 + 5 = 10$, etc. The statement is either true or false.

Definition 2: A theorem is a proposition that has been proved to be true e.g. the Pythagoras theorem, the Fermat's little theorem etc.

Definition 3: Reasoning can be viewed as drawing of inferences or conclusions from known or assumed facts. Deductive reasoning is the type of logic (science of correct reasoning that is fundamental to critical thinking and problem solving) that involves the application of a general statement to a specific case i.e. from general to specific. On the other hand, inductive reason is drawing conclusions from a specific case i.e. from specific to general.

Example 1: An example of deductive reasoning (from GENERAL to SPECIFIC) is the use of a formula to solve a particular problem e.g. we use the general quadratic formula to solve specific problems involving quadratic equations.

Example 2. An example of an inductive reasoning (from SPECIFIC to GENERAL) is where we use a SAMPLE to make conclusions or generalization of the entire POPULATION.

Definition 4: An argument is a structure that consists of PREMISES or ASSUMPTIONS and a CONCLUSION.

Example1: *If it rains Otieno will go to the market. It rained. Therefore, Otieno did not go to the market.*

This is an argument with two assumptions: If it rains Otieno will go to the market, and It rained. The conclusion is: Otieno did not go to the market.

Definition 5: A proof is an argument that is used to establish the truth of a theorem.

Definition 6: There exists different methods of mathematical proofs for establish the truth value of theorems. Generally there exists direct and indirect methods of proof that can be used under varying circumstances.

In methods of direct proof, one starts with the hypothesis of an implication and then shows that the conclusion is true. Some important methods of indirect proofs include proof by contradiction, proof by contrapositive, and proof by mathematical induction.

Definition 7: (An inductive set) A set X is called an inductive set if it satisfies the following properties;

(i) $1 \in X$

(ii) If $x \in X$, then $(x+1) \in X$

Definition 8: A Real number $\mathbb{R}$ is called a natural number $\mathbb{N} = \{1,2,3,4,\cdots\}$ if it belongs to every inductive set. The set of positive integers $\mathbb{Z}^+ = \{1,2,3,4,\cdots\}$ is also an inductive set.

Definition 9: Principle of mathematical induction

Let $p(n)$ be a proposition i.e. a statement that is either true or false, for each positive integer (or natural number n. If the following two conditions are satisfied;

(i) $p(1)$ is true, and

(ii) Whenever for $n = k$, $p(k)$ is true implies $p(k + 1)$ is true.

Then $p(n)$ is true for all positive integers n.

Steps to follow when proving by mathematical induction

Two unique steps to follow:

<u>Step 1:</u> Show by actual substitution that the proposed theorem or formula is true for some natural number n. For instances as $n = 1$, or $n = 2$, or $n = 3$ and so on.

<u>Step 2:</u> Assume that the proposed theorem or formula is true for $n = k$ (where k is an arbitrary positive integer or natural number). Then prove that is true for $n = k + 1$.

Then *conclude that the proposed theorem or formula is true for all positive integers or natural numbers greater than or equal to the n selected in step 1.*

Example 1: Prove that for all positive integers n then;

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Proof:

Step 1: Is the formula true for $n = 1$?

For $n = 1$ we have $\frac{1}{1\cdot 2}=\frac{1}{1+1}=\frac{1}{2}$. Hence it is true for $p(1)$.

Remark : We can check for $n = 2$ i.e. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{2}{2+1}=\frac{2}{3}$

The LHS: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{1}{2}+\frac{1}{6}=\frac{4}{6}=\frac{2}{3}$

For $n = 3$ we have the LHS as $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}=\frac{9}{12}=\frac{3}{4}$

This is equal to the RHS; $\frac{n}{n+1}=\frac{3}{3+1}=\frac{3}{4}$. *Clearly the LHS = RHS.*

Step 2: We assume that the formula is true for $n = k$.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}$$

Then we prove that it is true for $p(k+1)$, we add $(k+1)$ term to both sides i.e.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

We now simplify the RHS to get;

$$\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)(k+1)}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

Now $\frac{k+1}{k+2}$ is the value of $\frac{n}{n+1}$ when n is replaced with $(k+1)$. Therefore we can conclude that the formula is true for all positive integers.

Example 2: Prove that for all positive integers n then;

$$3^0+3^1+3^2+3^3+\cdots+3^{n-1}=\frac{3^n-1}{2}$$

Proof:

Step 1: Is the formula true for $n=1$?

For $n=1$ we have; $3^0=\frac{3^1-1}{2}\Rightarrow 1=\frac{2}{2}=1$. The LHS = RHS. Hence it is true for $p(1)$.

Again we can check for $n=2$; the LHS we shall have:- $3^0+3^1=4$.

On the RHS we have $\frac{3^2-1}{2}=\frac{8}{2}=4$. Hence it is true for $p(2)$.

Step 2: We assume that the formula is true for $n=k$ i.e.

$$3^0+3^1+3^2+3^3+\cdots+3^{k-1}=\frac{3^k-1}{2}$$

Then we prove that it is true for $p(k+1)$, we add $(k+1)^{th}$ term to both sides i.e.

$$3^0+3^1+3^2+3^3+\cdots+3^{k-1}+3^k=\frac{3^k-1}{2}+3^k$$

Next we simplify the RHS i.e.

$$\frac{3^k-1}{2}+3^k=\frac{3^k-1+2\cdot 3^k}{2}=\frac{3\cdot 3^k-1}{2}=\frac{3^{k+1}-1}{2}$$

Now $\left(\frac{3^{k+1}-1}{2}\right)$ is the value of $\left(\frac{3^n-1}{2}\right)$ when n is replaced with $(k+1)$.

Therefore we can conclude that the formula is true for all positive integers.

Example 3: Prove that $n^2 + n$ is even i.e. it is divisible by 2.

Proof:

Step 1: Is the theorem true for $n = 1$? $n^2 + 1 = 1^2 + 1 = 2$. Indeed it is true for $p(1)$.

Step 2: We assume it is true for $n = k$ i.e.

$$u_k = k^2 + k \text{ is even}$$

Next we show for $(k+1)^{th}$term i.e.

$$u_{k+1} = (k+1)^2 + (k+1)$$

Hence we have;

$$u_{k+1} - u_k = [(k+1)^2 + (k+1)] - [k^2 + k]$$

Working with the RHS we get;

$$= (k+1)(k+2) - k(k+1)$$

$$= (k+1)(k+2-k)$$

$$= 2(k+1) - \text{which is divisible 2}$$

$$\Rightarrow u_{k+1} = u_k + 2(k+1)$$

If u_k is divisible by 2 so is u_{k+1}.

Hence if the result is true for $n = k$ and it is true for $n = 1$ and $n = k + 1$, then by induction it is true for all positive integers n i.e. $n^2 + n$ is divisible by 2 (i.e. it is even).

Example 4: Show that $4^n - 1$ is divisible by 3.

Proof:

Step 1: It holds for $n = 1$ since $u_1 = 4^1 - 1 = 3$ and is divisible by 3.

Step 2: Assume it holds for $n = k$ i.e. $u_k = 4^k - 1$ is divisible by 3.

Next for $(k+1)$ we have $u_{k+1} = 4^{k+1} - 1$.

We get; $u_{k+1} - u_k = (4^{k+1} - 1) - (4^k - 1) = 4 \cdot 4^k - 1 - 4^k + 1 = 4 \cdot 4^k - 4^k = 4^k(4-1)$
$= 3 \cdot 4^k$ which is divisible by 3 $\Rightarrow u_{k+1} = u_k + 3 \cdot 4^k$

If u_k is divisible by 3 so is u_{k+1}.

Hence if the result is true for $n = k$ and it is true for $n = 1$ and $n = k + 1$, then by induction it is true for all positive integers n i.e. $4^n - 1$ is divisible by 3.

Exercise

1) Prove that for all positive integers n then;

a) $a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r-1}, r \neq 1$

b) $1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}, x \neq 1$

c) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

d) $2^0 + 2 + 2^3 + 2^4 + \cdots + 2^{n-1} = 2^n - 1$

e) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} = \frac{2n+1}{n+2}$

f) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n}{3}(n+1)(n+2)$

2) Prove that $n^3 - n$ is divisible by 6.

3) Prove that $n(n+1)(n+2)$ is divisible by 6.

4) Prove that $8^n + 6$ is divisible by 14.

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