Automatic Control Systems

Lecture-8

Analyze Response of a First Order System

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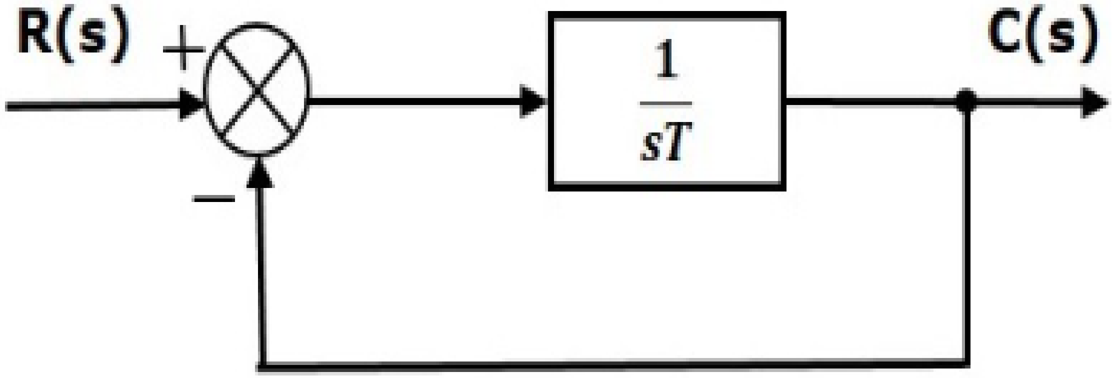
Session Objectives

By the end of this session, learners will be able to:

- Determine response of a first order system under different input signals
- Solve problems related to first order systems

Response of the First Order System

Consider the following block diagram of the closed loop control system. Its open loop transfer function, $\frac{1}{sT}$ is connected with a unity negative feedback.



Response of the First Order System

We know that the transfer function of the closed loop control system has unity negative feedback as,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Substitute, $G(s) = \frac{1}{sT}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{sT}}{1+\frac{1}{sT}} = \frac{1}{sT+1}$$

Response of the First Order System

- The power of s is one in the denominator term. Hence, the above transfer function is of the first order and the system is said to be the **first order system**.
- We can re-write the above equation as

$$C(s) = \left(\frac{1}{sT+1}\right) R(s)$$

Where,

C(s) is the Laplace transform of the output signal c(t),
R(s) is the Laplace transform of the input signal r(t), and
T is the time constant.

Response of the First Order System

Follow these steps to get the response (output) of the first order system in the time domain.

Take the Laplace transform of the input signal r(t)

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute $R(s)$ value in the above equation.

Do partial fractions of $C(s)$ if required.

Apply inverse Laplace transform to $C(s)$

Response of the First Order System

In the previous chapter, we have seen the standard test signals like

- ✓impulse
- ✓Step
- ✓ramp and
- ✓parabolic

Let us now find out the responses of the first order system for each input, one by one.

The name of the response is given as per the name of the input signal.

For example, the response of the system for an impulse input is called as impulse response

Impulse Response of First Order System

Consider the **unit impulse signal** as an input to the first order system.

So, $r(t) = \delta(t)$

Apply Laplace transform on both the sides

$$R(s) = 1$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = 1$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)(1) = \frac{1}{sT+1}$$

Impulse Response of First Order System

Rearrange the above equation in one of the standard forms of Laplace transforms.

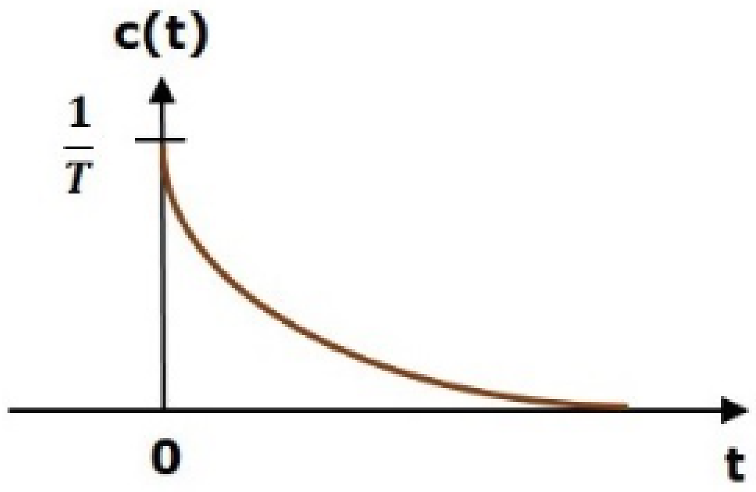
$$C(s) = \frac{1}{T\left(s + \frac{1}{T}\right)} \Rightarrow C(s) = \frac{1}{T}\left(\frac{1}{s + \frac{1}{T}}\right)$$

Apply inverse Laplace transform on both sides.

$$c(t) = \frac{1}{T} e^{\left(-\frac{t}{T}\right)} u(t)$$

Impulse Response of First Order System

The unit impulse response is shown in the following figure



The **unit impulse response**, c(t) is an exponential decaying signal for positive values of 't' and it is zero for negative values of 't'.

Step Response of First Order System

Consider the **unit step signal** as an input to first order system.

So, $r(t) = u(t)$

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s}\right) = \frac{1}{s\left(sT+1\right)}$$

Step Response of First Order System

Do partial fractions of C(s)

$$C(s) = \frac{1}{s(sT+1)} = \frac{A}{s} + \frac{B}{sT+1}$$

$$\Rightarrow \frac{1}{s(sT+1)} = \frac{A(sT+1)+Bs}{s(sT+1)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1 = A(sT+1) + Bs$$

Step Response of First Order System

By equating the constant terms on both the sides, you will get A = 1.

Substitute, A = 1 and equate the coefficient of the **s** terms on both the sides.

$$0 = T + B \Rightarrow B = -T$$

Substitute, A = 1 and B = −T in partial fraction expansion of $C(s)$

$$C(s) = \frac{1}{s} - \frac{T}{sT+1} = \frac{1}{s} - \frac{T}{T\left(s+\frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

Step Response of First Order System

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\left(\frac{t}{T}\right)}\right) u(t)$$

The **unit step response**, c(t) has both the transient and the steady state terms.

The transient term in the unit step response is

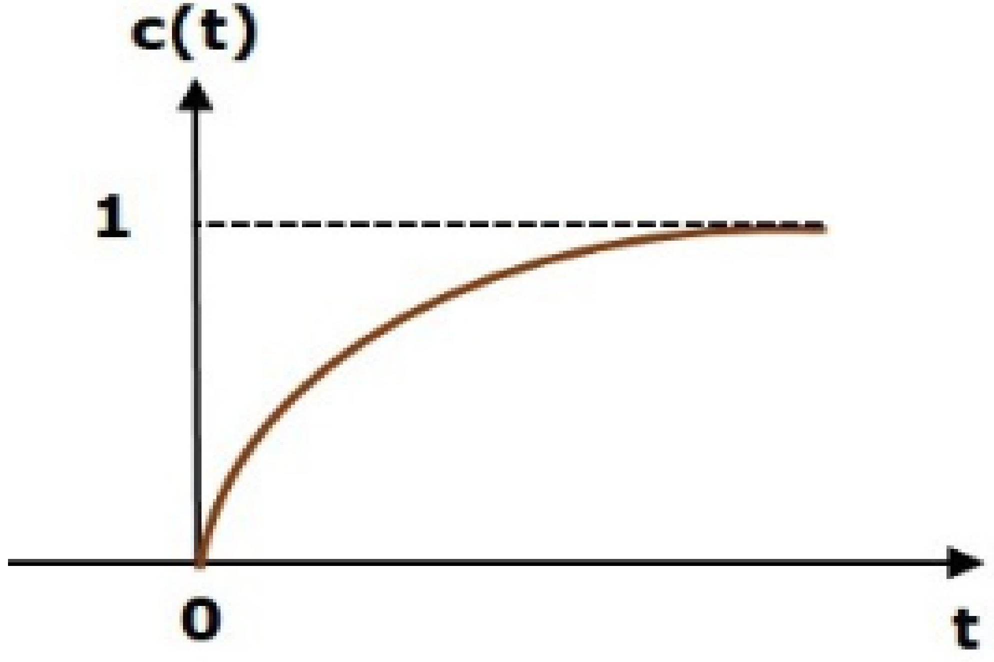
$$c_{tr}(t) = -e^{-\left(\frac{t}{T}\right)} u(t)$$

The steady state term in the unit step response is

$$c_{ss}(t) = u(t)$$

Step Response of First Order System

The following figure shows the unit step response.



The value of the **unit step response, c(t)** is zero at t = 0 and for all negative values of t.

It is gradually increasing from zero value and finally reaches to one in steady state. So, the steady state value depends on the magnitude of the input.

Ramp Response of First Order System

Consider the **unit ramp signal** as an input to the first order system.

$$So, r(t) = tu(t)$$

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s^2}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = \frac{1}{s^2}$ in the above equation.

Ramp Response of First Order System

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^2}\right) = \frac{1}{s^2(sT+1)}$$

Do partial fractions of $C(s)$

$$C(s) = \frac{1}{s^2(sT+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{sT+1}$$

$$\Rightarrow \frac{1}{s^2(sT+1)} = \frac{A(sT+1) + Bs(sT+1) + Cs^2}{s^2(sT+1)}$$

Ramp Response of First Order System

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1 = A(sT + 1) + Bs(sT + 1) + Cs^2$$

By equating the constant terms on both the sides, you will get A = 1

Substitute, A = 1 and equate the coefficient of the s terms on both the sides.

$$0 = T + B \Rightarrow B = -T$$

Ramp Response of First Order System

Similarly, substitute B = −T and equate the coefficient of s^2 terms on both the sides. You will get

$$C = T^2 .$$

Substitute A = 1, B = −T and $C = T^2$ in the partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{sT+1} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{T\left(s+\frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+\frac{1}{T}}$$

Ramp Response of First Order System

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(t - T + Te^{-\left(\frac{t}{T}\right)}\right)u(t)$$

The **unit ramp response**, c(t) has both the transient and the steady state terms.

The transient term in the unit ramp response is

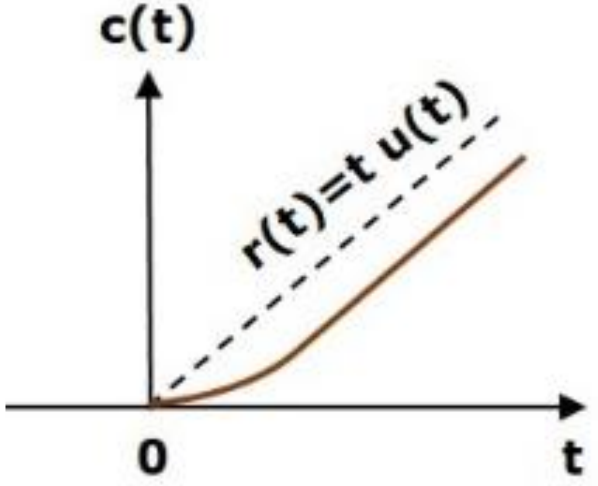
$$c_{tr}(t) = Te^{-\left(\frac{t}{T}\right)}u(t)$$

The steady state term in the unit ramp response is

$$c_{ss}(t) = (t - T)u(t)$$

Ramp Response of First Order System

The following figure shows the unit ramp response.



The **unit ramp response**, c(t) follows the unit ramp input signal for all positive values of t. But, there is a deviation of T units from the input signal.

Parabolic Response of First Order System

Consider the **unit parabolic signal** as an input to the first order system.

So, $r(t) = \frac{t^2}{2}u(t)$

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s^3}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right)R(s)$

Substitute $R(s) = \frac{1}{s^3}$ in the above equation.

Parabolic Response of First Order System

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^3}\right) = \frac{1}{s^3(sT+1)}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{1}{s^3(sT+1)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{sT+1}$$

After simplifying, you will get the values of A, B, C and D as 1, $-T, T^2 \; and \; -T^3$ respectively

Substitute these values in the above partial fraction expansion of C(s).

Parabolic Response of First Order System

$$C(s) = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^3}{sT+1} \Rightarrow C(s) = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^2}{s+\frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(\frac{t^2}{2} - Tt + T^2 - T^2 e^{-\left(\frac{t}{T}\right)}\right) u(t)$$

The **unit parabolic response**, c(t) has both the transient and the steady state terms.

The transient term in the unit parabolic response is

$$C_{tr}(t) = -T^2 e^{-\left(\frac{t}{T}\right)} u(t)$$

Parabolic Response of First Order System

The steady state term in the unit parabolic response is

$$C_{ss}(t) = \left(\frac{t^2}{2} - Tt + T^2\right) u(t)$$

From these responses, we can conclude that the first order control systems are not stable with the ramp and parabolic inputs because these responses go on increasing even at infinite amount of time.

Parabolic Response of First Order System

The first order control systems are stable with impulse and step inputs because these responses have bounded output. But, the impulse response doesn’t have steady state term. So, the step signal is widely used in the time domain for analyzing the control systems from their responses.

Solved Example 1

A casual system having the transfer function $H(s) = \frac{1}{s+2}$ is excited with $10u(t)$. The time at which the output reaches 99% of its steady state value is

(a) 2.7 sec

(b) 2.5 sec

(c) 2.3 sec

(d) 2.1 sec

Solution

$$H(s) = \frac{1}{s+2}$$

$$r(t) = 10\mu(t)$$

$$R(s) = \frac{10}{s}$$

$$C(s) = H(s)\,R(s) = \frac{10}{s(s+2)}$$

$$C(s) = \frac{5}{s} - \frac{5}{s+2}$$

$$C(t) = 5[1 - e^{-2t}]$$

Steady state value = 5

99% of the steady value reaches at

Solution cont'

$$5[1 - e^{-2t}] = \frac{5\times99}{100}$$

$$5(1 - e^{-2t}) = 5 \times 0.99$$

$$1 - e^{-2t} = 0.99$$

Or $e^{-2t} = 0.1$

$$-2t = l_n(0.1)$$

$$t = 2.3\ sec$$

Option (c)

Solved example 2

The unit impulse response of a linear time invariant system is the unit step function u(t). For t > 0, the response of the system to an excitation $e^{-at}u(t), a > 0$ will be

(a) ae^{-at}

(b) $(1/a)(1-e^{-at})$

(c) $a(1-e^{-at})$

(d) $1-e^{-at}$

Solution

$$H(s) = \frac{1}{s}$$

System excitation $r(t) = e^{-at}\, u(t)$

$$R(s) = \frac{1}{s+a}$$

Response of the system $C(s) = R(s)\, H(s) = \frac{1}{s(s+a)}$

$$C(s) = \frac{1}{a}\left[\frac{1}{s} - \frac{1}{(s+a)}\right]$$

$$C(t) = \frac{1}{a}\left[1 - e^{-at}\right]$$

Option (b)

References

1. Smarajit Gosh (2007), Control systems, Pearson Education
2. Manirakiza and Kanyarwanda (2020), ELT 303Lecture Note, IPRC Gishari.
3. K. Ogata (1997), Modern Control Engineering, 3rd Edition, Prentice Hall.

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