

Automatic Control Systems

Lecture-9

Analyze Response of a Second Order System

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Session Objectives

By the end of this session learners will be able to:

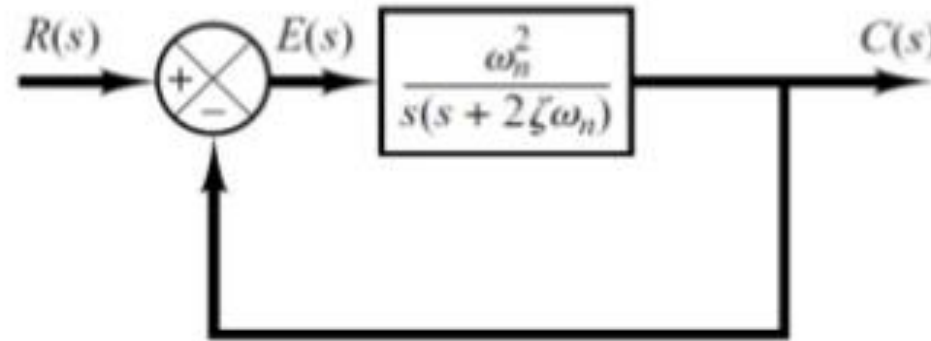
- Determine the response of a second order system
- Classify second order systems based on the value of damping ratio
- Explain time domain specifications of a second order system
- Determine the steady state error of a second order system

Response of Second Order System

Introduction

- A general second-order system is characterized by the following transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



ω_n \longrightarrow **un-damped natural frequency** of the second order system, which is the frequency of oscillation of the system without damping.

ζ \longrightarrow **damping ratio** of the second order system, which is a measure of the degree of resistance to change in the system output.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

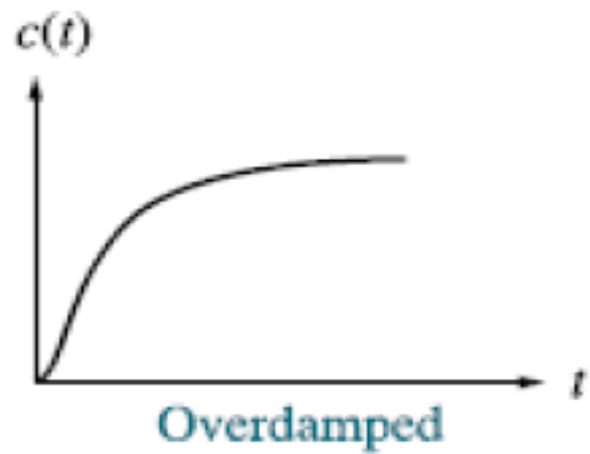
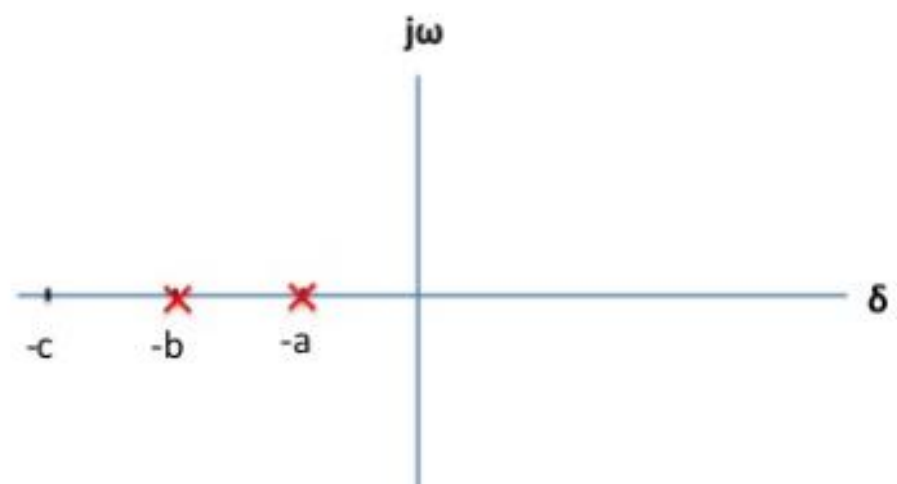
Two poles of the system are

$$s = \frac{-2\delta\omega_n \pm \sqrt{(2\delta\omega_n)^2 - 4\omega_n^2}}{2} = \frac{-2(\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1})}{2}$$

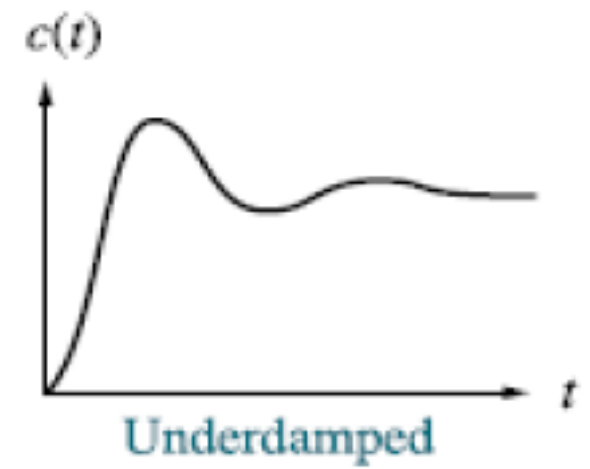
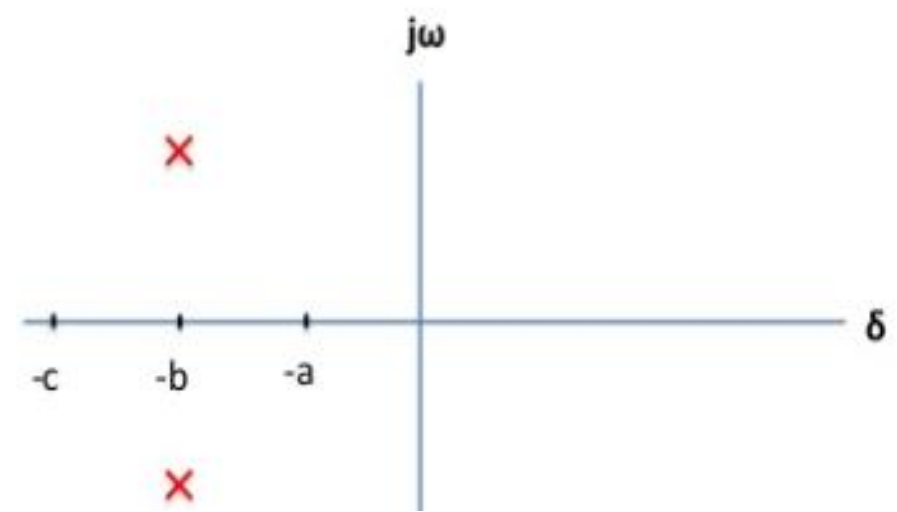
$$\mathbf{s} = \begin{array}{l} -\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1} \\ -\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1} \end{array}$$

Depending upon the value of ζ , there are four cases

1. *Overdamped* - when the system has two real distinct poles ($\zeta > 1$).



2. *Underdamped* - when the system has two complex conjugate poles ($0 < \zeta < 1$)



3. *Undamped* - when the system has two imaginary poles ($\zeta = 0$).



4. *Critically damped* - when the system has two real but equal poles ($\zeta = 1$).



Step Response of Second Order System

(1) *Underdamped case* ($0 < \zeta < 1$): In this case, $C(s)/R(s)$ can be written

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

We can modify the denominator term of the transfer function as follows –

$$\begin{aligned} s^2 + 2\delta\omega_n s + \omega_n^2 &= \{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\} + \omega_n^2 - (\delta\omega_n)^2 \\ &= (s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2) \end{aligned}$$

The transfer function becomes,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)} \right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)} \right) \left(\frac{1}{s} \right) = \frac{\omega_n^2}{s((s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2))}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s((s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2))} = \frac{A}{s} + \frac{Bs + C}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}$$

After simplifying, you will get the values of A, B and C as 1 , -1 and $-2\delta\omega_n$ respectively. Substitute

these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s} - \frac{s + 2\delta\omega_n}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{s + \delta\omega_n}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)} - \frac{\delta\omega_n}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)} \quad C(s) = \frac{1}{s} - \frac{(s + \delta\omega_n)}{(s + \delta\omega_n)^2 + (\omega_n\sqrt{1 - \delta^2})^2} - \frac{\delta}{\sqrt{1 - \delta^2}} \left(\frac{\omega_n\sqrt{1 - \delta^2}}{(s + \delta\omega_n)^2 + (\omega_n\sqrt{1 - \delta^2})^2} \right)$$

Substitute, $\omega_n\sqrt{1 - \delta^2}$ as ω_d in the above equation.

$$C(s) = \frac{1}{s} - \frac{(s + \delta\omega_n)}{(s + \delta\omega_n)^2 + \omega_d^2} - \frac{\delta}{\sqrt{1 - \delta^2}} \left(\frac{\omega_d}{(s + \delta\omega_n)^2 + \omega_d^2} \right)$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\delta\omega_n t} \cos(\omega_d t) - \frac{\delta}{\sqrt{1 - \delta^2}} e^{-\delta\omega_n t} \sin(\omega_d t) \right) u(t)$$

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1 - \delta^2}} \left((\sqrt{1 - \delta^2}) \cos(\omega_d t) + \delta \sin(\omega_d t) \right) \right) u(t)$$

If $\sqrt{1 - \delta^2} = \sin(\theta)$, then ' δ ' will be $\cos(\theta)$. Substitute these values in the above equation.

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1 - \delta^2}} (\sin(\theta) \cos(\omega_d t) + \cos(\theta) \sin(\omega_d t)) \right) u(t)$$

$$\Rightarrow c(t) = \left(1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1 - \delta^2}} \right) \sin(\omega_d t + \theta) \right) u(t)$$

(2) *Critically damped case* ($\zeta = 1$): If the two poles of $C(s)/R(s)$ are equal, the system is said to be a critically damped one.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s + \omega_n)^2} \right) R(s)$$

$$C(s) = \left(\frac{\omega_n^2}{(s + \omega_n)^2} \right) \left(\frac{1}{s} \right) = \frac{\omega_n^2}{s(s + \omega_n)^2}$$

$$C(s) = \frac{\omega_n^2}{s(s + \omega_n)^2} = \frac{A}{s} + \frac{B}{s + \omega_n} + \frac{C}{(s + \omega_n)^2}$$

After simplifying, you will get the values of A, B and C as 1, -1 and $-\omega_n$ respectively. Substitute

these values in the above partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2}$$

$$c(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}) u(t)$$

3. **Undamped** - when the system has two imaginary poles ($\zeta = 0$).

$$C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2} \right) \left(\frac{1}{s} \right) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)}$$

$$c(t) = (1 - \cos(\omega_n t)) u(t)$$

Overdamped - when the system has two real distinct poles ($\zeta > 1$).

We can modify the denominator term of the transfer function as follows –

$$s^2 + 2\delta\omega_n s + \omega_n^2 = \{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\} + \omega_n^2 - (\delta\omega_n)^2$$

$$= (s + \delta\omega_n)^2 - \omega_n^2 (\delta^2 - 1) \quad \frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \delta\omega_n)^2 - \omega_n^2 (\delta^2 - 1)}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s + \delta\omega_n)^2 - \omega_n^2 (\delta^2 - 1)} \right) R(s)$$

$$C(s) = \left(\frac{\omega_n^2}{(s + \delta\omega_n)^2 - (\omega_n \sqrt{\delta^2 - 1})^2} \right) \left(\frac{1}{s} \right) = \frac{\omega_n^2}{s(s + \delta\omega_n + \omega_n \sqrt{\delta^2 - 1})(s + \delta\omega_n - \omega_n \sqrt{\delta^2 - 1})}$$

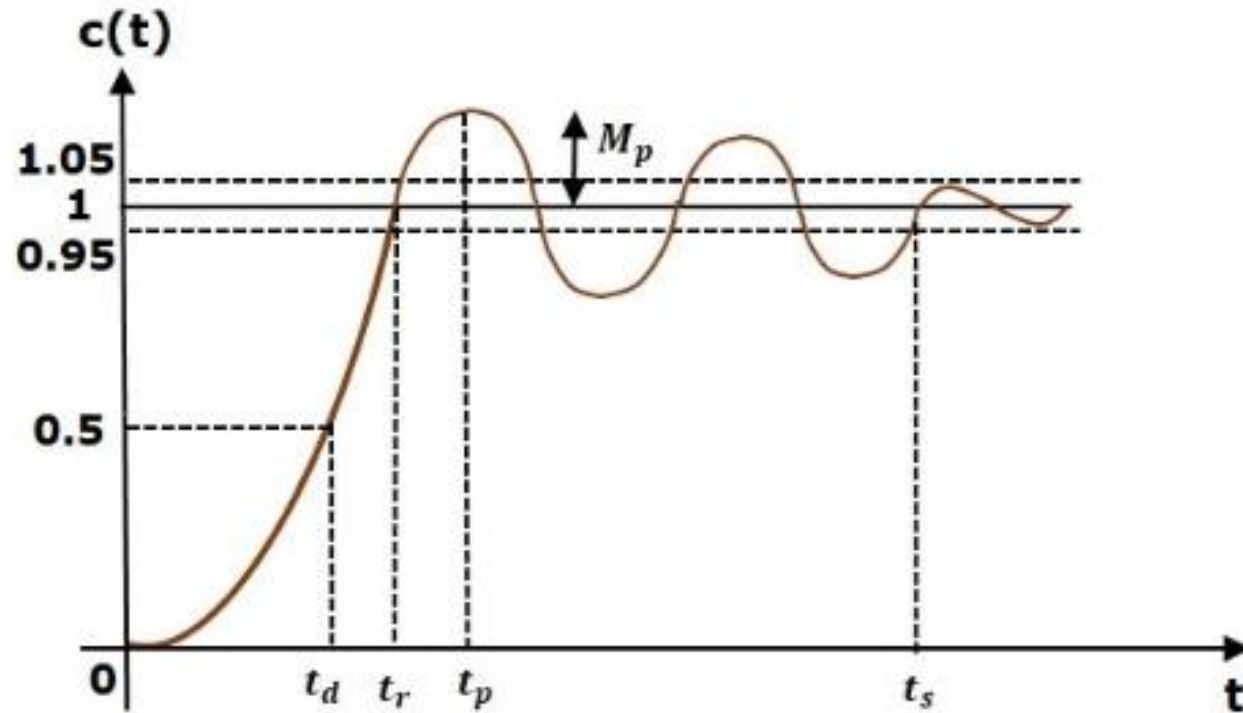
$$= \frac{A}{s} + \frac{B}{s + \delta\omega_n + \omega_n\sqrt{\delta^2 - 1}} + \frac{C}{s + \delta\omega_n - \omega_n\sqrt{\delta^2 - 1}}$$

After simplifying, you will get the values of A, B and C as 1, $\frac{1}{2(\delta + \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})}$ and

$$\frac{-1}{2(\delta - \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})} \text{ respectively. } C(s) = \frac{1}{s} + \frac{1}{2(\delta + \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})} \left(\frac{1}{s + \delta\omega_n + \omega_n\sqrt{\delta^2 - 1}} \right) - \left(\frac{1}{2(\delta - \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})} \right) \left(\frac{1}{s + \delta\omega_n - \omega_n\sqrt{\delta^2 - 1}} \right)$$

$$c(t) = \left(1 + \left(\frac{1}{2(\delta + \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})} \right) e^{-(\delta\omega_n + \omega_n\sqrt{\delta^2 - 1})t} - \left(\frac{1}{2(\delta - \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})} \right) e^{-(\delta\omega_n - \omega_n\sqrt{\delta^2 - 1})t} \right)$$

Time Domain Specification of Second Order System



1. **DELAY TIME (t_d):** The delay time is the time required for the response to reach 50% of the final value in first time.

2. **RISE TIME (t_r):** It is time required for the response to rise from 10% to 90% of its final value for over-damped systems and 0 to 100% for under-damped systems.

$$t_r = \frac{\pi - \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n \sqrt{1-\xi^2}}$$

3. **PEAK TIME (t_p):** The peak time is the time required for the response to reach the first peak of the time response or first peak overshoot.

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}$$

4. MAXIMUM OVERSHOOT (M_p):

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\left(\omega_n \sqrt{1-\xi^2}\right)t + \phi\right]$$

Maximum overshoot occur at peak time, $t=t_p$

Put,

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}$$

$$M_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$$

$$M_p \% = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} * 100$$

5. SETTLING TIME (t_s): The settling time is defined as the time required for the transient response to reach and stay within the prescribed percentage error.

Time consumed in exponential decay up to 98% of the input. The settling time for a second order system is approximately four times the time constant 'T'.

$$t_s = 4T = \frac{4}{\xi\omega_n}$$

EXAMPLE 1: The open loop transfer function of a servo system with unity feedback is given by

$$G(s) = \frac{10}{(s+2)(s+5)}$$

Determine the damping ratio, undamped natural frequency of oscillation. What is the percentage overshoot of the response to a unit step input.

SOLUTION: Given that

$$G(s) = \frac{10}{(s+2)(s+5)}$$

$$H(s) = 1$$

Characteristic equation $1 + G(s)H(s) = 0$

$$1 + \frac{10}{(s+2)(s+5)} = 0$$

$$s^2 + 7s + 20 = 0$$

Compare with $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$ We get

$$\omega_n^2 = 20$$

$$2\xi\omega_n = 7$$

$$\therefore \omega_n = \sqrt{20} = 4.472 \text{ rad/sec}$$

$$2 * \xi * 4.472 = 7$$

$$\xi = 0.7826$$

$$M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} = e^{-\frac{\pi * 0.7826}{\sqrt{1-(0.7826)^2}}} * 100 = 1.92\%$$

$$\omega_n = 4.472 \text{ rad/sec}$$

$$\xi = 0.7826$$

$$M_p = 1.92\%$$

EXAMPLE 2: A feedback system is described by the following transfer function

$$G(s) = \frac{12}{s^2 + 4s + 16}$$

$$H(s) = Ks$$

The damping factor of the system is 0.8. determine the overshoot of the system and value of 'K'.

SOLUTION: We know that

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$\frac{C(s)}{R(s)} = \frac{12}{s^2 + (4 + 16K)s + 16}$$

$s^2 + (4 + 16K)s + 16 = 0$ is the characteristic eqn.

Compare with

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$\omega_n^2 = 16 \quad \therefore \omega_n = 4 \text{ rad/sec.}$$

$$2\xi\omega_n = 4 + 16K$$

$$2 * 0.8 * 4 = 4 + 16K \quad \therefore K = 0.15$$

$$M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} * 100 = e^{-\frac{\pi 0.8}{\sqrt{1-(0.8)^2}}} * 100$$

$$M_p = 1.5\%$$

Example 3

Consider a system with the transfer function $(s) = \frac{s+6}{Ks^2+s+6}$. Its damping ratio will be 0.5 when the value of K is

(a) $2/6$

(b) 3

(c) $1/6$

(d) 6

Soln. Transfer function $G(s) = \frac{s+6}{ks^2+s+6}$

Damping ratio $\xi = 0.5$

Comparing with 2nd order equation $s^2 + 2 \xi \omega_n s + \omega_n^2$

$$G(s) = \frac{s+6}{k \left[s^2 + \frac{s}{k} + \frac{6}{k} \right]}$$

$$\omega_n^2 = \frac{6}{k}, \quad \omega_n = \sqrt{\frac{6}{k}}$$

$$2 \xi \omega_n = \frac{1}{k}$$

$$2 \xi \sqrt{\frac{6}{k}} = \frac{1}{k}$$

$$2 \times 0.5 \sqrt{\frac{6}{k}} = \frac{1}{k}$$

$$\text{or } \frac{6}{k} = \frac{1}{k^2}$$

$$\text{or } k = \frac{1}{6}$$

Option (c)

Steady State Errors

The deviation of the output of control system from desired response during steady state is known as steady state error. It is represented as e_{ss} . We can find steady state error using the final value theorem as follows.

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$$

Where,

$E(s)$ is the Laplace transform of the error signal, $e(t)$

Let us discuss how to find steady state errors for unity feedback and non-unity feedback control systems one by one.

Steady State Errors for Unity Feedback Systems

Consider the following block diagram of closed loop control system, which is having unity negative feedback



Where,

- $R(s)$ is the Laplace transform of the reference input signal $r(t)$
- $C(s)$ is the Laplace transform of the output signal $c(t)$

Steady State Errors for Unity Feedback Systems

We know the transfer function of the unity negative feedback closed loop control system as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \quad \Rightarrow \quad C(s) = \frac{R(s)G(s)}{1 + G(s)}$$

The output of the summing point is -

$$E(s) = R(s) - C(s)$$

Substitute $C(s)$ value in the above equation.

Steady State Errors for Unity Feedback Systems

$$E(s) = R(s) - \frac{R(s)G(s)}{1 + G(s)} \quad \Rightarrow \quad E(s) = \frac{R(s) + R(s)G(s) - R(s)G(s)}{1 + G(s)}$$

$$\Rightarrow E(s) = \frac{R(s)}{1 + G(s)}$$

Substitute $E(s)$ value in the steady state error formula

$$e_{ss} = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

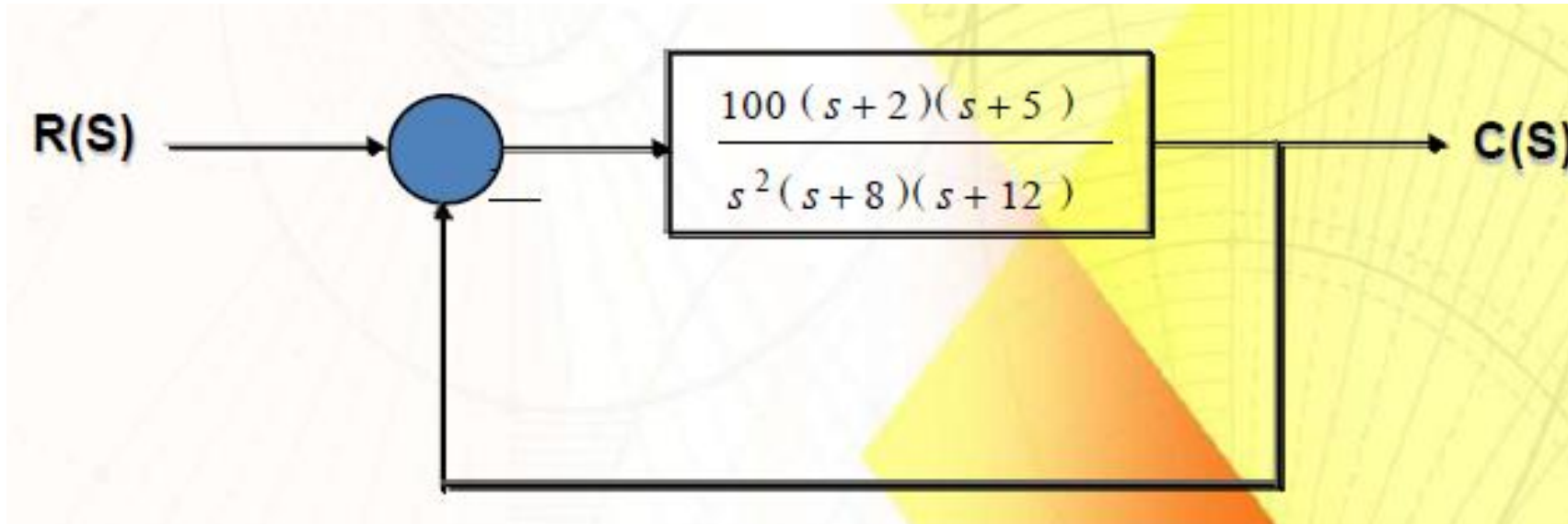
The following table shows the steady state errors and the error constants for standard input signals like unit step, unit ramp & unit parabolic signals

Input signal	Steady state error e_{ss}	Error constant
unit step signal	$\frac{1}{1+k_p}$	$K_p = \lim_{s \rightarrow 0} G(s)$
unit ramp signal	$\frac{1}{K_v}$	$K_v = \lim_{s \rightarrow 0} sG(s)$
unit parabolic signal	$\frac{1}{K_a}$	$K_a = \lim_{s \rightarrow 0} s^2G(s)$

Where, K_p , K_v and K_a are position error constant, velocity error constant and acceleration error constant respectively.

Example

For the system shown in figure below evaluate the static error constants and find the expected steady state errors for the standard step, ramp and parabolic inputs.



Solution

$$G(s) = \frac{100(s+2)(s+5)}{s^2(s+8)(s+12)}$$

$$K_p = \lim_{s \rightarrow 0} G(s)$$

$$K_p = \lim_{s \rightarrow 0} \left(\frac{100(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$K_p = \infty$$

$$K_v = \lim_{s \rightarrow 0} sG(s)$$

$$K_v = \lim_{s \rightarrow 0} \left(\frac{100s(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$K_v = \infty$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s)$$

$$K_a = \lim_{s \rightarrow 0} \left(\frac{100 s^2 (s+2)(s+5)}{s^2 (s+8)(s+12)} \right)$$

$$K_a = \left(\frac{100 (0+2)(0+5)}{(0+8)(0+12)} \right) = 10.4$$

$$K_p = \infty \quad K_v = \infty \quad K_a = 10.4$$

$$e_{ss} = \frac{1}{1 + K_p} = 0$$

$$e_{ss} = \frac{1}{K_v} = 0$$

$$e_{ss} = \frac{1}{K_a} = 0.09$$

References

1. Benjamin C. Kuo (1975), Automatic Control Systems, 3rd Edition, Prentice Hall.
2. K. Ogata (1997), Modern Control Engineering, 3rd Edition, Prentice Hall.
3. Smarajit Gosh (2007), Control systems, Pearson Education.

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