

Automatic Control Systems

Lecture-10

Analyze System Stability Using Routh Hurwitz Technique

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STABILITY OF A CONTROL SYSTEM

Specific Objectives

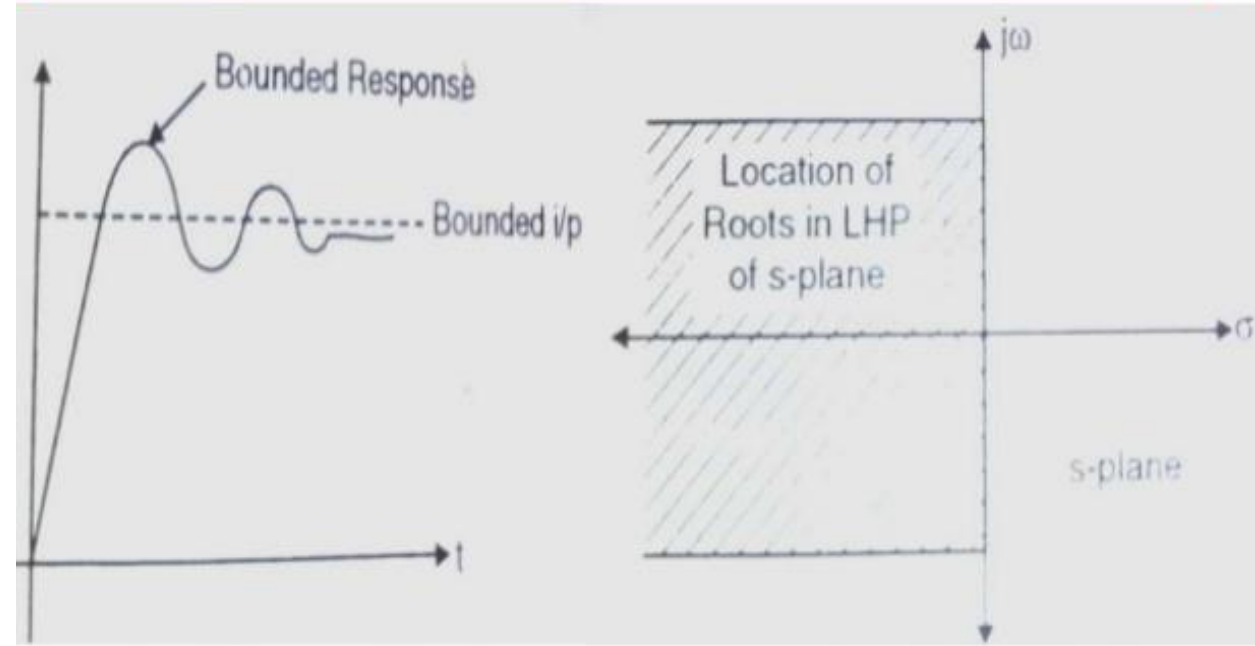
- Appreciate the importance of stability
- Analyze different types of stability
- Apply Routh's stability criterion for stability analysis and solve the numerical.

Stable System

A linear time invariant system is stable if following conditions are satisfied:

- A bounded input is given to the system, the response of the system is bounded and controllable.
- In the absence of the inputs, the output should tend to zero as time increases.

Stable System



Bounded i/p bounded o/p
for stable system

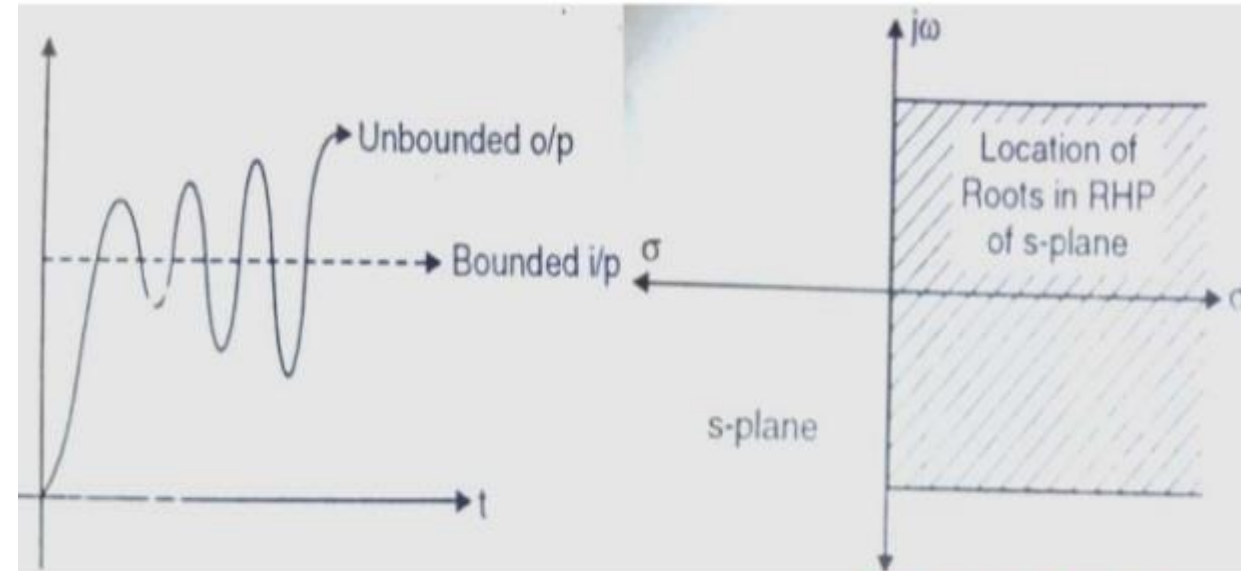
Location of roots
for stable system

Jongeun Choi, ME 451 Control Systems Lecture notes, Michigan State University, Slide 19. Available at:
https://www.egr.msu.edu/classes/me451/jchoi/2012/notes/ME451_L9_Stability.pdf

Unstable System

- A linear time invariant system comes under the class of unstable system if the system is excited by a bounded input, response is unbounded.
- This means once any input is given system output goes on increasing & designer does not have any control on it

Unstable System



Bounded i/p Unbounded o/p
for unstable system

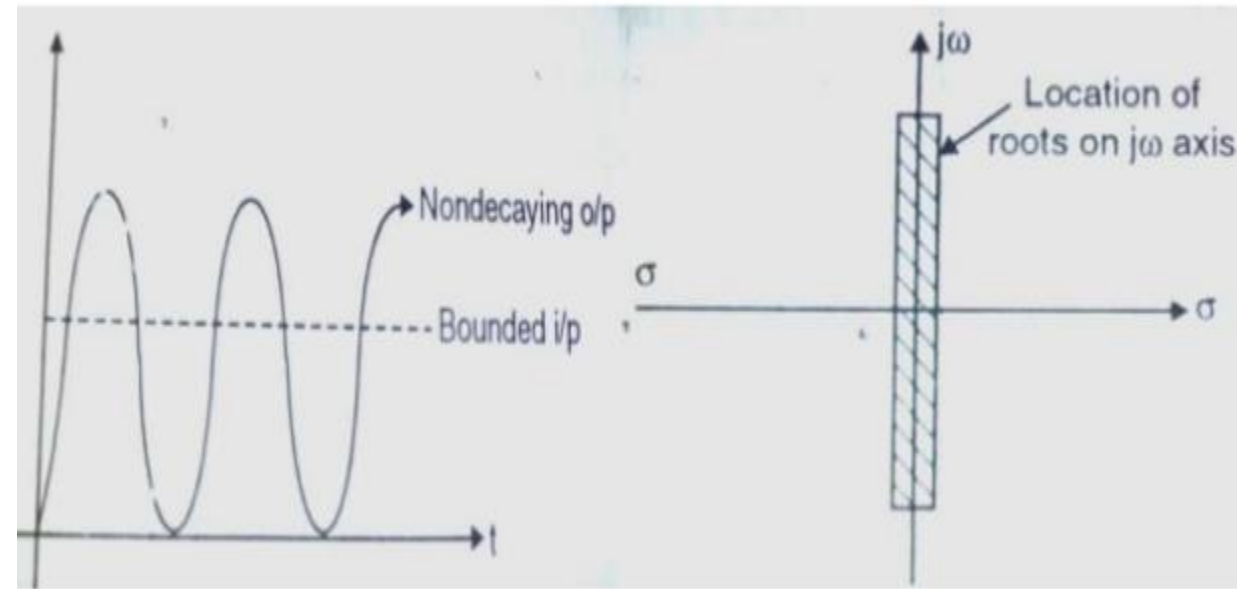
Location of roots
for unstable system

Jongeun Choi, ME 451 Control Systems Lecture notes, Michigan State University, Slide 6. Available at:
https://www.egr.msu.edu/classes/me451/jchoi/2012/notes/ME451_L9_Stability.pdf

Critically Stable System

- When the input is given to a linear time invariant system, for critically stable systems the output does not go on increasing infinitely nor does it go to zero as time increases.
- The output usually oscillates in a finite range or remains steady at some value.
- Such systems are not stable as their response does not decay to zero. Neither they are defined as unstable because their output does not go on increasing infinitely.

Critically stable System



Bounded i/p & o/p response
for critically stable system

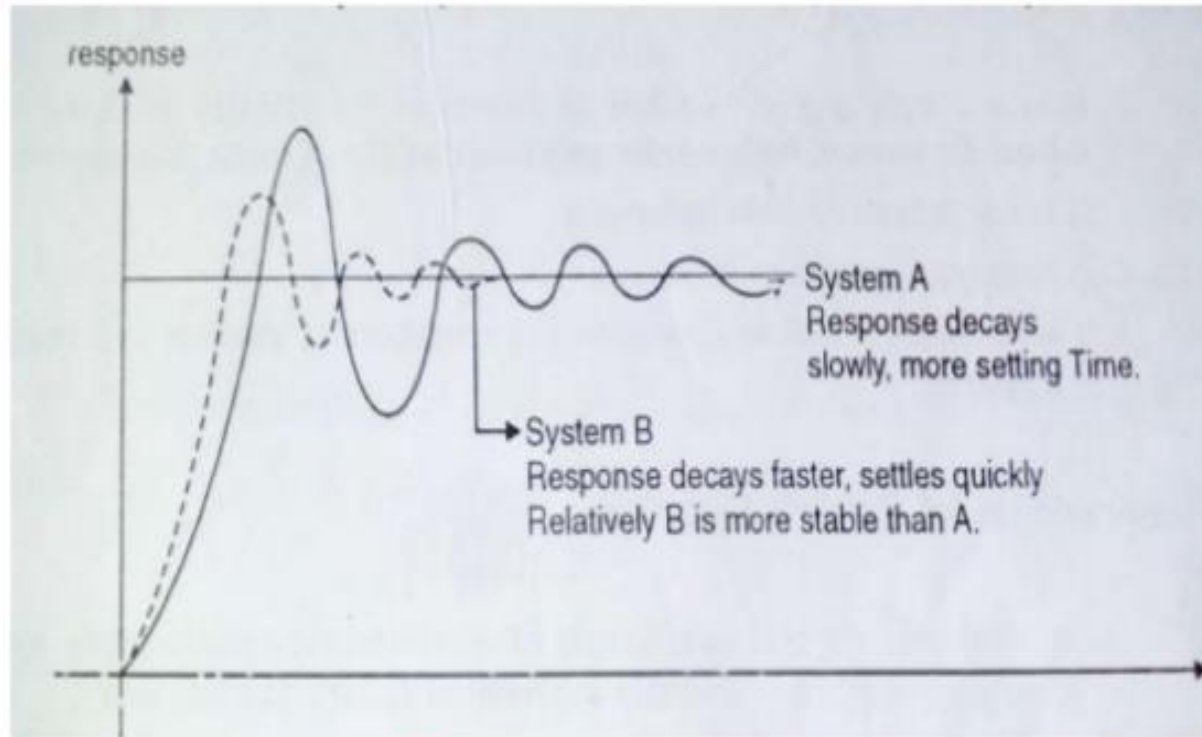
Location of roots
for critically stable system

Jongeun Choi, ME 451 Control Systems Lecture notes, Michigan State University, Slide 6. Available at:
https://www.egr.msu.edu/classes/me451/jchoi/2012/notes/ME451_L9_Stability.pdf

Relative Stability

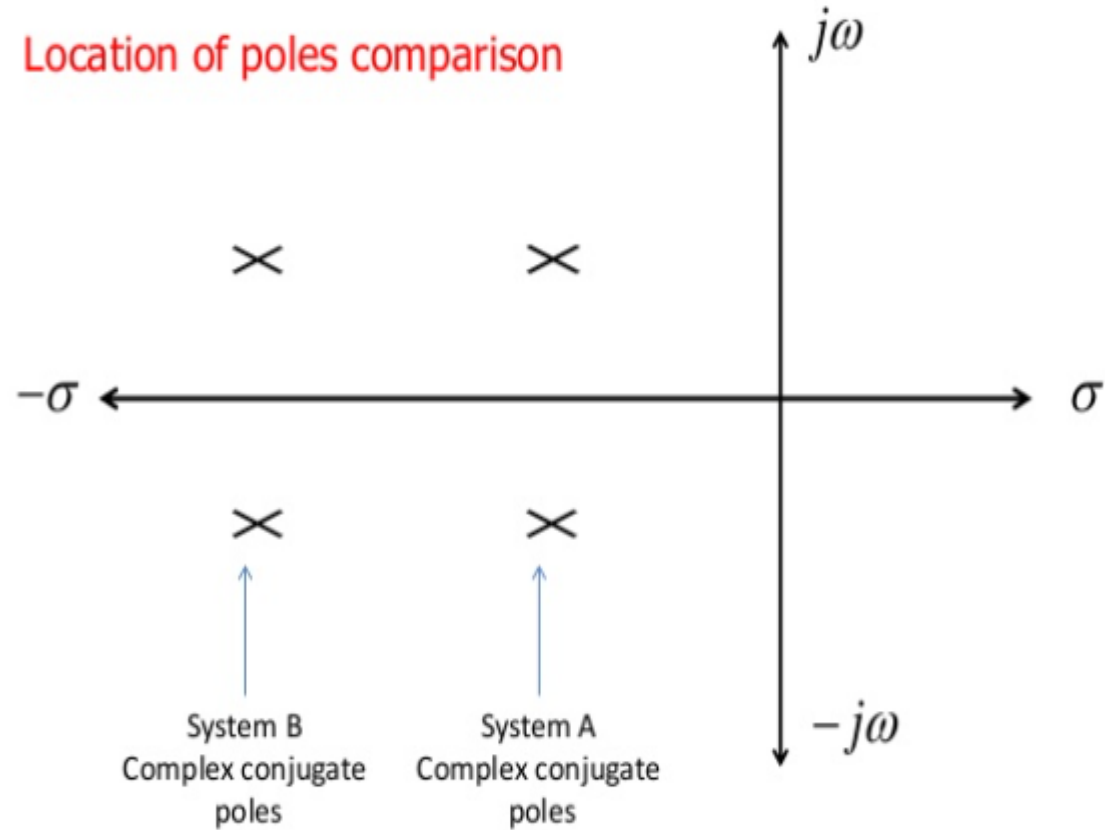
- A system may be absolutely stable i.e. it may have passed the Routh stability test.
- As a result their response decays to zero under zero input conditions.
- The ratio at which these decay to zero is important to check the concept of “Relative stability”
- When the poles are located far away from $j\omega$ axis in LHP of s -plane, the response decays to zero much faster, as compared to the poles close to $j\omega$ -axis.
- The more the poles are located far away from $j\omega$ -axis the more is the system relatively stable.

Relative Stability



Response comparison

Relative Stability



Sobieraj, S.; Sieklucki, G.; Gromba, J. Relative Stability of Electrical into Mechanical Conversion with BLDC Motor-Cascade Control. *Energies* 2021, 14, 704. <https://doi.org/10.3390/en14030704>, page 12.

Routh's Stability Criterion

For the transfer function;

$$\frac{C(s)}{R(s)} = \frac{b_0s^m + b_1s^{m-1} + \dots + b_m}{a_0s^n + a_1s^{n-1} + \dots + a_n}$$

In this criterion, the coefficients of denominator are arranged in an Array called "Routh's Array";

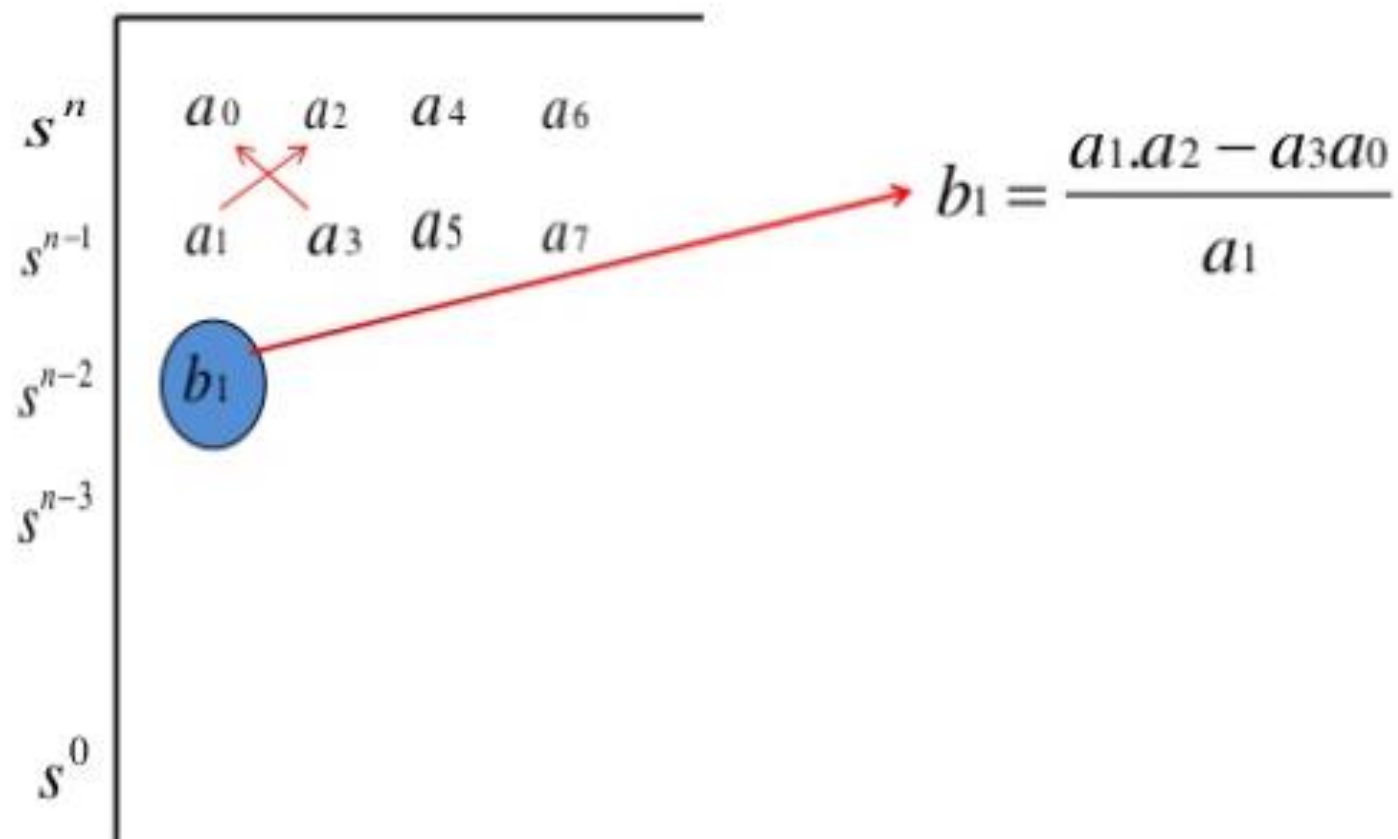
$$a_0s^n + a_1s^{n-1} + \dots + a_n = 0$$

The coefficients of s^n and s^{n-1} row are directly written from the given equation.

$$a_0s^n + a_1s^{n-1} + \dots + a_n = 0$$

The Routh's array as below;

For next row i.e. s^{n-2} ;



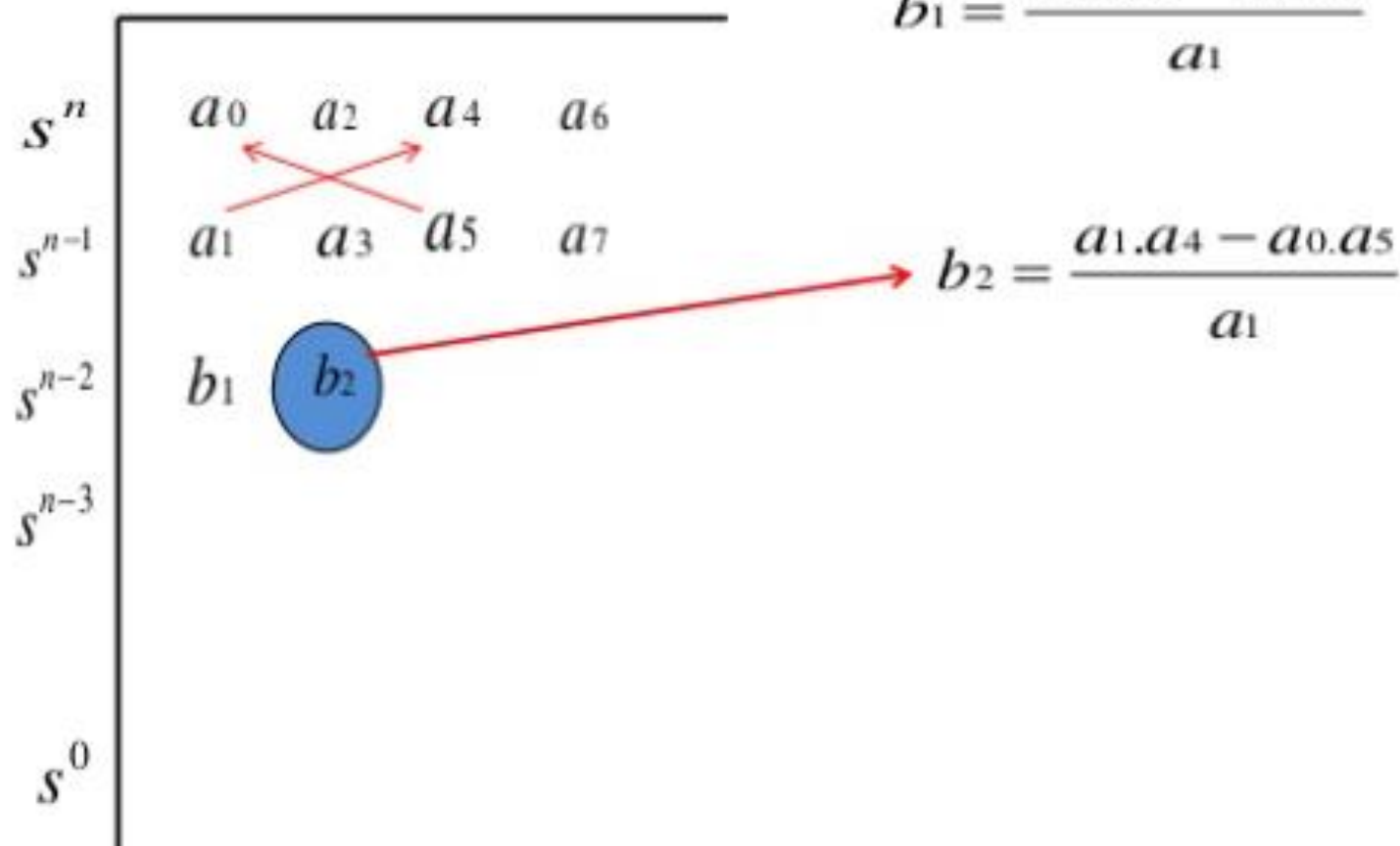
The coefficients of s^n and s^{n-1} row are directly written from the given equation.

$$a_0s^n + a_1s^{n-1} + \dots + a_n = 0$$

The Routh's array as below;

For next row i.e. s^{n-2} ;

$$b_1 = \frac{a_1 \cdot a_2 - a_3 a_0}{a_1}$$

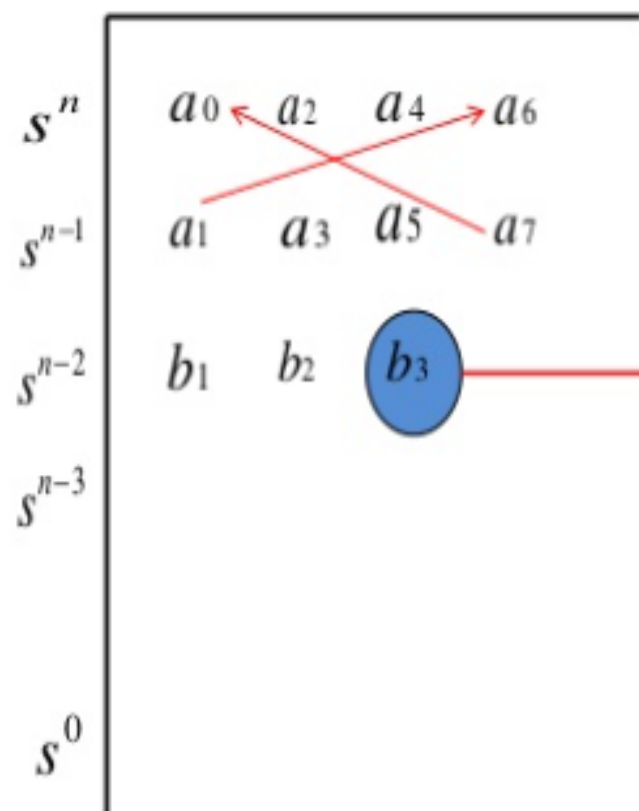


The coefficients of s^n and s^{n-1} row are directly written from the given equation.

$$a_0s^n + a_1s^{n-1} + \dots + a_n = 0$$

The Routh's array as below;

For next row i.e. s^{n-2} ;



$$b_1 = \frac{a_1 \cdot a_2 - a_3 a_0}{a_1}$$

$$b_2 = \frac{a_1 \cdot a_4 - a_0 \cdot a_5}{a_1}$$

$$b_3 = \frac{a_1 \cdot a_6 - a_0 \cdot a_7}{a_1}$$

Now the same technique is used, for the next row i.e. s^{n-3} row, but only previous two rows are used i.e. s^{n-1} and s^{n-2}

The Routh's array as below;

s^n	a_0	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}	b_1	b_2	b_3	
s^{n-3}	c_1			
s^0				

For next row i.e. s^{n-2} ;

$$b_1 = \frac{a_1 \cdot a_2 - a_3 a_0}{a_1}$$

$$b_2 = \frac{a_1 \cdot a_4 - a_0 \cdot a_5}{a_1}$$

$$b_3 = \frac{a_1 \cdot a_6 - a_0 \cdot a_7}{a_1}$$

For next row i.e. s^{n-3} ;

$$c_1 = \frac{b_1 \cdot a_3 - b_2 a_1}{b_1}$$

The Routh's array as below;

s^n	a_0	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}	b_1	b_2	b_3	
s^{n-3}	c_1	c_2		
s^0				

$$b_1 = \frac{a_1 \cdot a_2 - a_3 a_0}{a_1}$$

$$b_2 = \frac{a_1 \cdot a_4 - a_0 \cdot a_5}{a_1}$$

$$b_3 = \frac{a_1 \cdot a_6 - a_0 \cdot a_7}{a_1}$$

For next row i.e. s^{n-3}

$$c_1 = \frac{b_1 \cdot a_3 - b_2 a_1}{b_1}$$

$$c_2 = \frac{b_1 \cdot a_5 - b_3 a_1}{b_1}$$

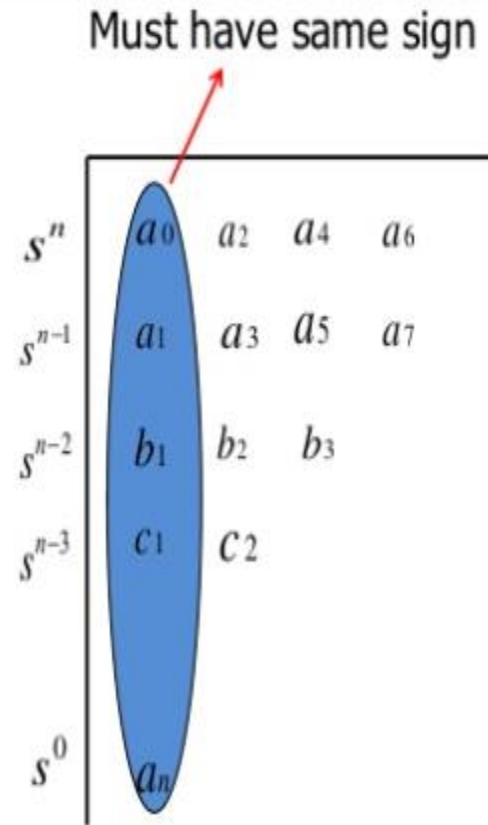
➤ Each column will reduce by one as we move down the array.

➤ This process is obtained till last row is obtained.

s^n	a_0	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}	b_1	b_2	b_3	
s^{n-3}	c_1	c_2		
s^0	a_n			

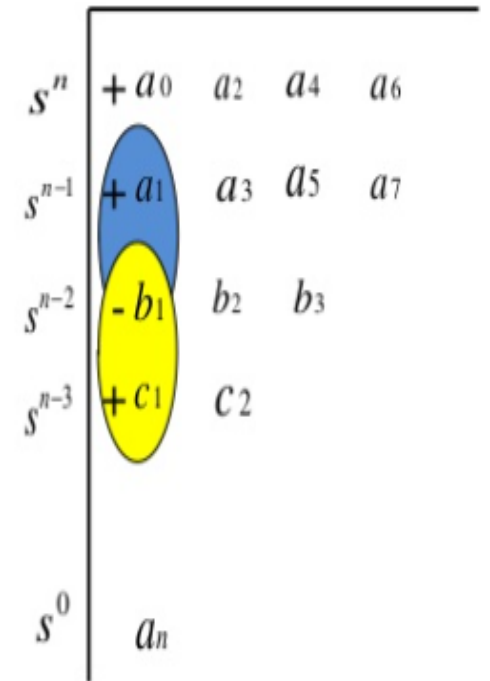
Routh's Criterion

- The necessary & sufficient conditions for a system to be stable is all terms in the first column at Routh's Array should have same sign.
- There should not be any sign change in first column.



Routh's Criterion

- When there are sign changes in the first column of Routh's array then the system is unstable.
- There are roots in RHP.
- The number of sign changes equal the number of roots in RHP.



Example 1

Comment on stability. $s^4 + 2s^3 + 6s^2 + 10s + 3 = 0$

s^4	1	6	3
s^3	2	10	0
s^2	b_1	b_2	
s^1	c_1		
s^0	a_n		

$$b_1 = \frac{a_1 a_2 - a_3 a_0}{a_1}$$

$$c_1 = \frac{b_1 a_3 - b_2 a_1}{b_1}$$

$$c_1 = \frac{1 \times 10 - 2 \times 3}{1}$$

$$c_1 = 4$$

$$b_1 = \frac{(2 \times 6) - (1 \times 10)}{2}$$

$$a_n = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$a_n = \frac{4 \times 3 - 1 \times 0}{4}$$

$$b_2 = \frac{2 \times 3 - 1 \times 0}{2}$$

$$a_n = 3$$

$$b_2 = 3$$

Example 1

cont....

s^4	1	6	3
s^3	2	10	0
s^2	1	3	
s^1	4		
s^0	3		

As no sign change in first column; system is stable

Example 2

Comment on stability.

$$2s^3 + 4s^2 + 4s + 12 = 0$$

s^3	2	4	0
s^2	4	12	
s^1	b_1	b_2	
s^0	a_n		

$$b_1 = \frac{a_1 a_2 - a_3 a_0}{a_1}$$

$$a_n = \frac{(-2 \times 12) - (4 \times 0)}{-2}$$

$$b_1 = \frac{(4 \times 4) - (2 \times 12)}{4}$$

$$a_n = 12$$

$$b_1 = -2$$

$$b_2 = \frac{(4 \times 0) - (2 \times 0)}{4}$$

$$b_2 = 0$$

Example 2

cont....

s^3	2	4	0
s^2	4	12	
s^1	-2	0	
s^0	12		

**There are two sign changes
+4 to -2 and -2 to +12.
Hence two roots are in RHP
S-plane and system is unstable**

Example 3

Comment on stability. $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$

s^5	1	2	3
s^4	1	2	5
s^3	b_1	b_2	
s^2	c_1	c_2	
s^1	d_1		
s^0	a_n		

$$b_1 = \frac{(1 \times 2) - (2 \times 1)}{1}$$

$$b_1 = 0$$

$$b_2 = \frac{(1 \times 3) - (5 \times 1)}{1}$$

$$b_2 = -2$$

s^5	1	2	3
s^4	1	2	5
s^3	0	-2	
s^2	c_1	c_2	
s^1	d_1		
s^0	a_n		

Case I – “When the first of any row is zero and the rest of the row is non-zero.”

Routh's Criterion Special Cases

Case I – “When the first of any row is zero and the rest of the row is non-zero.” Here the next row cannot be formed as division by 0 will take place.

Method to Overcome: A method to overcome above problem is to replace s by $\frac{1}{z}$ and complete the Routh's test for z .

Example 3

cont..

Replace s by (1/z)

$$\left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^4 + 2\left(\frac{1}{z}\right)^3 + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right) + 5 = 0$$

Take L.C.M

$$\frac{1 + z + 2z^2 + 2z^3 + 3z^4 + 5z^5}{z^5} = 0$$

$$1 + z + 2z^2 + 2z^3 + 3z^4 + 5z^5 = 0$$

$$5z^5 + 3z^4 + 2z^3 + 2z^2 + z + 1 = 0$$

Use above characteristics equation and complete Routh's Test

z^5	5	2	1
z^4	3	2	1
z^3	$-\frac{4}{3}$	$-\frac{2}{3}$	
z^2	$\frac{1}{2}$	1	
z^1	2		
z^0	1		

There are two sign changes in first column.

Hence two roots are in RHP S-plane and system is unstable

Example 4

Comment on stability. $s^4 + 6s^3 + 11s^2 + 6s + 10 = 0$

s^4	1	11	10
s^3	6	6	0
s^2	b_1	b_2	
s^1	c_1		
s^0	a_n		

$$b_1 = \frac{(6 \times 11) - (1 \times 6)}{6}$$

$$b_1 = 10$$

$$b_2 = \frac{(6 \times 10) - (1 \times 0)}{6}$$

$$b_2 = 10$$

$$c_1 = \frac{(10 \times 6) - (10 \times 6)}{10} = 0$$

s^4	1	11	10
s^3	6	6	0
s^2	10	10	
s^1	0		
s^0	a_n		

Case II – “When all elements in any one row is zero.”

Routh's Criterion Special Cases

Case II – “When all elements in any one row is zero.”

Method to Overcome:

- ✓ Here form an auxillary equation with the help of the coefficients of the coefficients of the row just above the row of zeros.
- ✓ Take the derivative of this equation and replace it's coefficients in the present row of zeros.
- ✓ Then proceed for Routh's test.

Example 4

cont....

s^4	1	11	10
s^3	6	6	0
s^2	10	10	
s^1	0		
s^0	a_n		

Here s^1 row breaks down.
Hence write auxiliary equation
for s^2 .

$$A(s) = 10s^2 + 10$$

(Note each term of next column
differs by degree of 2)

Take derivative of auxiliary equation

$$\frac{d}{ds} A(s) = 20s$$

Use these for s row coefficients.

s^4	1	11	10
s^3	6	6	0
s^2	10	10	
s^1	20		
s^0	a_n		

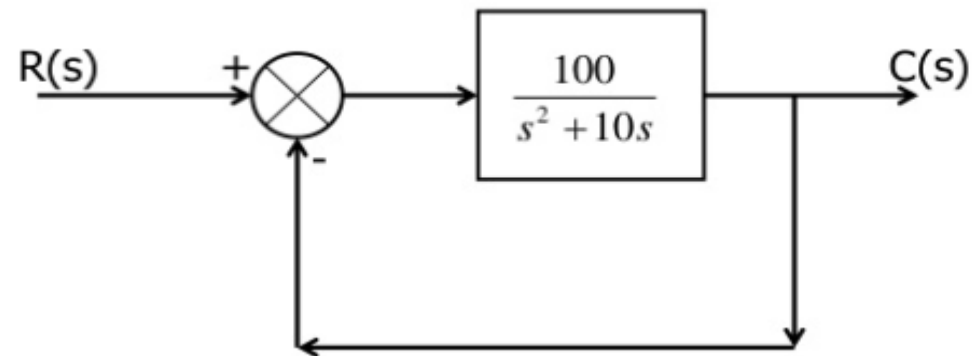
$$a_n = \frac{(20 \times 10) - (10 \times 0)}{20}$$

$$a_n = 10$$

s^4	1	11	10
s^3	6	6	0
s^2	10	10	
s^1	20		
s^0	10		

As no sign change in first column; system is stable

Problem: Using routh's criteria find the stability for given figure.



Application of Routh's Criterion

✓ The gain is kept in terms of k and Routh's array is solved to find k for stable operation.

Example 5

Determine the range of k for stable system. $s^4 + 5s^3 + 5s^2 + 4s + k = 0$

s^4	1	5	k
s^3	5	4	0
s^2	4.2	k	
s^1	$\frac{16.8 - 5k}{4.2}$		
s^0	k		

Example 5

cont.....

For stability all elements of first column 1 should be positive

$$\text{i.e. } k > 0 \quad \text{For } s^0 \text{ row} \quad \text{-----(1)}$$

$$\text{and } \frac{16.8 - 5k}{4.2} > 0 \quad \text{For } s^1 \text{ row}$$

$$\text{i.e. } 16.8 > 5k \text{ or } k < \frac{16.8}{5} \quad \text{-----(2)}$$
$$k < 3.36$$

Thus combining equations (1) and (2), $0 < k < 3.36$

This is the range of k stable operation.

Example 6

Determine the range of k for stable system.

$$s^4 + 4s^3 + 4s^2 + 3s + k = 0$$

s^4	1	2	k
s^3	4	3	
s^2	$5/4$	k	
s^1	$\frac{15-16k}{5}$		
s^0	k		

For the system to be stable, all elements of column 1 should be positive

i.e $k > 0$

For s^0 row.....(1)

and $\frac{15-16k}{5}$

For s^1

$15-16k > 0$ or $K < 15/16$

$K < 0.9375$

..... (2)

Thus, combining equations (1) and (2), $0 < k < 0.9375$

This is the range of k stable operation.

Advantages of Routh's Criterion

- It is a simple algebraic method to determine the stability of closed loop without solving for roots of higher order polynomial of the characteristics equation.
- It is not tedious or time consuming.
- It progress systematically.
- It is frequently used to determine the conditions of absolute & relative stability of a system.
- It can determine range of k for stable operation.

References

1. Smarajit Gosh (2007), Control systems, Pearson Education
2. Benjamin C. Kuo (1975), Automatic Control Systems, 3rd Edition, Prentice Hall.

THANK YOU FOR YOUR KIND ATTENTION