Mathematics for Science

Solutions to the Examination

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Section A (Compulsory)

a) Convert the rectangular coordinates (3, -4) to polar coordinates. (2 mks)

$$x = 3, y = -4 \text{ hence } \tan\theta = -\frac{4}{3} => \theta = \tan^{-1} -\frac{4}{3} \therefore \theta \approx -53.1^{o}$$

$$r = \sqrt{3^2 + 4^2} = 5$$

Polar coordinates $(5, -53.1^{o})$

b) Distinguish between an ellipse and a circle (without a diagram). (2 mks)

A circle is a locus of all points on a plane equidistant from a fixed point while an ellipse is the locus of all points in a plane such that the sum of the distances from two fixed points the foci, to any other point on the locus is constant.

c) Convert the following polar equation into rectangular equation; $r = Cos\theta + 2Sin\theta$, and hence identify the graphs of the polar equation and the resultant equation. (4 mks)

Multiply both sides by r to get $r^2 = rcos\theta + 2rsin\theta$ hence we have;

$$x^2 + y^2 = x + 2y => x^2 + y^2 - x - 2y = 0$$

<u>Graphs</u>

- Resultant graph

$$b^2 - 4ac = 0 - 4(1)(1) = -4 < 0 \text{, since } b = 0 \text{ and } a = c \text{ this is a circle but centre is not origin.}$$

- Polar equation; circle

d) Find the numerical value of k if the expression $x^3 + kx^2 + 7x - 6$ has a reminder of – 4 when divided by $(x + 2)$. (3 mks)

$$f(-2) = (-2)^3 + 4k - 14 - 6 = -4$$

$$4k - 28 = -4$$

$$4k = 24 \therefore k = 6$$

e) Show that $(x - 2)$ is a factor of the expression $x^4 - 13x^2 + 36$. (2 mks)

$f(2) = 2^4 - 13(4) + 36 = 0 \Rightarrow (x - 2)$ is a factor.

f) A school committee of 9 members is to be constituted from 8 parents, 6 teachers and the headteacher. In how many ways can the committee be formed to include the headteacher and 5 parents? (2 mks)

$$8C3 \times 6C3 = \frac{8!}{5!\,3!} \times \frac{6!}{3!\,3!} = 1120$$

g) Determine the general equation of a circle which has a diameter with endpoints at $(0,5)$and $(8,5)$ (4 mks)

The radius is 4 and the centre is the point (4, 5) hence the standard equation is $(x-4)^2 + (y-5)^2 = 16$

The general equation is; $x^2 - 8x + 16 + y^2 - 10y + 25 - 16 = 0$

$$x^2 - 8x + y^2 - 10y + 25 = 0$$

h) Solve the following equation for the values of x between 0 and 100 degrees. (4 mks)

$$6 \sin x \cos x = 1$$

$$3(2\, sinx \cos x) = 1$$

$$\sin 2x = \frac{1}{3}$$

$$=> 2x = \sin^{-1}\left(\frac{1}{3}\right) \approx 19.47^o, 160.53^o, 379.47^o, 520.53^o$$

$$\therefore x \approx 9.73^o, 80.26^o, 189.73^o, 260.53^o$$

i) A parabolic signal satellite disk has the shape $x^2 = 9y.$ At what point should the signal receiver be located? (3 mks)

$$\mathrm{x}^2 = 4\mathrm{py} => 4\mathrm{p} = 9 \therefore \mathrm{p} = \frac{9}{4} = 2.25$$

The point is $(0, 2.25)$

j) Simplify; $\frac{\cos^2 x - 1}{\sin 2x}$ and hence determine acute angle x if $\frac{\cos^2 x - 1}{\sin 2x} = -2$ (4 mks)

$$\frac{\cos^2 \mathrm{x} - 1}{\sin 2\mathrm{x}} = \frac{-\sin^2 \mathrm{x}}{2\sin \mathrm{x}\cos\mathrm{x}} = \frac{-\sin \mathrm{x}}{2\cos \mathrm{x}} = -\frac{1}{2}\tan \mathrm{x}$$

Hence;

$$-\frac{1}{2}\tan x = -2 => \tan x = 4 \therefore x \approx 76^o$$

Section B

Question 1 (compulsory) – 20 marks

a) Determine the vertex, focus, directrix, and axis for parabola $y^2 - 4x + 10y + 13 = 0$ (4 mks)

$$y^2 + 10y + k = 4x - 13 + k$$

But $k = \left(\frac{10}{2}\right)^2 = (5)^2$

Hence we have;

$$(y + 5)^2 = 4x - 13 + 25$$
$$(y + 5)^2 = 4x + 12 = 4(x + 3)$$
$$(y + 5)^2 = 4(x + 3)$$

This is of the form;

$$(\mathrm{y} - \mathrm{k})^2 = 4\mathrm{p}(\mathrm{x} - \mathrm{h})$$
$$\text{Thus; } 4\mathrm{p} = 4 \therefore \mathrm{p} = 1$$
$$k = -5, h = -3$$

Vertex is the point (h, k) = (- 3, -5)

Axis is the line $y = -5$

Directrix is $x = h - p = -3 - 1 = -4 \; i.e. \, x = -4$

Focus is the point $(h + p, k) = (-2, -5)$

b) Find the partial fraction decomposition of; $\frac{2x^2+10x-3}{(x+1)(x^2-9)}$ (5 mks)

$$\frac{2x^2 + 10x - 3}{(x + 1)(x^2 - 9)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{x + 3}$$
$$2x^2 + 10x - 3 = A(x^2 - 9) + B(x + 1)(x + 3) + C(x + 1)(x - 3)$$

When $x = 3$ we get; $18 + 30 - 3 = 24B => B = \frac{15}{8}$

When $x = -3$ we get; $18 - 30 - 3 = 12C => -\frac{5}{4}$

When $x = -1$ we get; $2 - 10 - 3 = -8A => \frac{11}{8}$

$$\frac{2x^2 + 10x - 3}{(x + 1)(x^2 - 9)} = \frac{11}{7(x + 1)} + \frac{15}{8(x - 3)} - \frac{5}{4(x + 3)}$$

c) Determine the center and the radius of the circle equation; (4 mks)

$$x^2 + y^2 - 10x + 14y + 25 = 0$$

$$x^2 - 10x + k_1 + y^2 + 14y + k_2 = -25 + k_1 + k_2$$

$$k_1 = \left(-\frac{10}{2}\right)^2 = (-5)^2 ; k_2 = \left(\frac{14}{2}\right)^2 = (7)^2$$

$$(x - 5)^2 + (y + 7)^2 = -25 + 25 + 49 = 49$$

Centre (5, -7) , radius 7 units

d) Prove that; $\frac{\sin A}{\sin 2A}+\frac{\cos A}{1+\cos 2A}=\sec A.$ (4 mks)

$\frac{\sin A}{2\sin A\cos A}+\frac{\cos A}{1+\cos^2 A-\sin^2 A}$

$=\frac{\sin A}{2\sin A\cos A}+\frac{\cos A}{\cos^2 A+\cos^2 A}$ since $\cos^2 A+\sin^2 A=1$

$=\frac{1}{2\cos A}+\frac{\cos A}{2\cos^2 A}$

$=\frac{1}{2\cos A}+\frac{1}{2\cos A}=\frac{2}{2\cos A}=\frac{1}{\cos A}=\sec A$

e) Determine the numerical values of a and b if $(x-1)$ and $(x+2)$ are both factors of x^3+ax^2+bx-2 (3 mks)

$$f(1)=1+a+b-2=0 \Rightarrow a+b=1 \cdots \text{(i)}$$

$$f(-2)=-8+4a-2b-2=0 \Rightarrow 2a-b=5 \cdots \text{(ii)}$$

from (i) a = 1 – b then (ii) becomes;

$$2(1-b)-b=5$$

$$2-2b-b=5$$

$$-3b=3 \therefore b=-1$$

$a=1+1=2$

f) In the triangle ABC, $a = 4.73cm$, $b = 3.5cm$ and $C=42.2^\circ$ Calculate the size of angles A and B. (5 mks)

$$c^2=a^2+b^2-2ab\cos C$$

$$c^2=4.73^2+3.5^2-2(4.73)(3.5)\cos 42.2^\circ \approx 10.1$$

$$c \approx 3.18 \text{ cm}$$

$$\frac{4.73}{\sin A}=\frac{3.18}{\sin 42.2^\circ} \Rightarrow \sin A=\frac{4.73\sin 42.2^\circ}{3.18}\approx 0.9991 \Rightarrow A\approx 87.6^\circ$$

$$\frac{3.5}{\sin B}=\frac{3.18}{\sin 42.2^\circ} \Rightarrow \sin B=\frac{3.5\sin 42.2^\circ}{3.18}\approx 0.7393 \Rightarrow B\approx 47.67$$

g) A computer science student decided to generate some codes from the word DOUBLES.

(i) How many 3-letter code words can be formed from this word? (1 mk)

$$7P3=\frac{7!}{4!}=210$$

(ii) How many of these 3-letter codes contain the letter D? (2 mks)

$$3 \times 6P2 = 3 \times \frac{6!}{4!} = 90$$

(iii) How many of these 3-letter codes do not contain a vowel? (2 mks)

4 letters are consonants hence we have;

$$4P3 = \frac{4!}{1!} = 24$$

Question 2 (Optional) – 10 marks

a) Identify the conic section given the equation below. (2 mks)

$$2x^2 - y^2 - 7 = 0$$

$$b^2 - 4ac = 0 - 4(2)(-1) = 8 > 0$$

It is a hyperbola or a degenerate case.

b) Convert into rectangular equation the polar equation $r^2 = 4\sin 2\theta$. What name is given to the curve of the polar equation. (4 mks)

$$r^2 = 8\sin\theta cos\theta$$

$$x^2 + y^2 = 8 \cdot \frac{y}{r} \cdot \frac{x}{r} = \frac{8xy}{r^2} = \frac{8xy}{x^2 + y^2}$$

$$(x^2 + y^2)^2 - 8xy = 0$$

c) Write the equation of the hyperbola below in standard form. (4 mks)

$$25x^2 - 9y^2 - 100x - 72y - 269 = 0$$

$$25(x^2 - 4x) - 9(y^2 + 8y) = 269$$

$$25(x^2 - 4x + k_1) - 9(y^2 + 8y + k_2) = 269 + 25k_1 - 9k_2$$

$$k_1 = \left(-\frac{4}{2}\right)^2 = (-2)^2; k_2 = \left(\frac{8}{2}\right)^2 = (4)^2$$

$$25(x-2)^2 - 9(y+4)^2 = 269 + 25(4) - 9(16) = 225$$

$$\frac{(x-2)^2}{9} - \frac{(y+4)^2}{25} = 1$$

Question 3 (Optional) – 10 marks

a) How many selections of 4 letters can be made from the 9 letters? (2 mks)

$$\frac{9!}{5!\,4!} = 126$$

b) Prove that; (4 mks)

$$\frac{\sin^3\theta}{\cos^2\theta} + \sin\theta = \tan\theta\sec\theta$$

<u>LHS</u>

$$\frac{\sin^3\theta + \sin\theta\cos^2\theta}{\cos^2\theta}$$

$$= \frac{\sin\theta(\sin^2\theta + \cos^2\theta)}{\cos^2\theta}$$
$$= \frac{\sin\theta}{\cos^2\theta}$$
$$= \frac{\sin\theta}{\cos\theta\cos\theta}$$
$$= \frac{\sin\theta}{\cos\theta} \times \frac{1}{\cos\theta}$$

But

$\frac{\sin\theta}{\cos\theta} = \tan\theta$ and $\frac{1}{\cos\theta} = \sec\theta$ hence we have;

$$\frac{\sin\theta}{\cos\theta} \times \frac{1}{\cos\theta} = \tan\theta\sec\theta$$

c) Find all the values of x where $0^o \leq x \leq 360^o$ that satisfy the equation; (4 mks)

$$3\cos x - 5\sin 2x = 0$$
$$3\cos x = 5\sin 2x$$
$$3\cos x = 5 \cdot 2\sin x\cos x$$
$$3 = 10\sin x$$
$$\sin x = 0.3 \therefore x = \sin^{-1} 0.3 \approx 17.5^o, 162.5^o$$

Question 4 (Optional) – 10 marks

a) Find the remainder when the expression $x^3 - 4x^2 + 3x + 5$ is divided by $(x - 3)$ (1 mks)

$$f(3) = 3^3 - 36 + 9 + 5 = 5$$

b) Given that $(x + 2)$ and $(x - 3)$ are factors of $ax^3 + ax^2 + bx + 12$. Find the numerical values of a and b. (4 mks)

$$f(-2) = -8a + 4a - 2b + 12 = -4a - 2b = -12 \Rightarrow 2a + b = 6 \cdots (i)$$
$$f(3) = 27a + 9a + 3b + 12 = 36a + 3b = -12 \Rightarrow 12a + b = -4 \cdots (ii)$$
$$\text{from (i)} b = 6 - 2a \Rightarrow 12a + 6 - 2a = -4$$
$$10a = -10 \therefore a = -1 \text{ and hence } b = 6 + 2 = 8$$

c) Decompose into partial fractions; (5 mks)

$$\frac{x^2-6x+2}{x^2(x-2)^2} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

$$x^2 - 6x + 2 = A(x-2)^2 + Bx(x-2)^2 + Cx^2(x-2) + Dx^2$$
$$= Ax^2 - 4xA + 4A + Bx^3 - 4x^2B + 4Bx + Cx^3 - 2Cx^2 + Dx^2$$
$$= (B + C)x^3 + (A - 4B - 2C + D)x^2 + (4B - 4A)x + 4A$$

Hence;

$$B + C = 0 \cdots (i)$$
$$A - 4B - 2C + D = 1 \cdots (ii)$$

$$4B - 4A = -6 \cdots \text{(iii)}$$

$$4A = 2 \Rightarrow A = \frac{1}{2}$$

From (iii) $4B = 4A - 6 = 2 - 6 = -4 \therefore B = -1$

From (i) $C = -B = 1$

From (ii) $D = 1 + 4B + 2C - A = 1 + 4(-1) + 2(1) - \frac{1}{2} = -\frac{3}{2}$

$$\frac{x^2 - 6x + 2}{x^2(x-2)^2} = \frac{1}{2x^2} - \frac{1}{x} + \frac{1}{x-2} - \frac{3}{2(x-2)^2}$$