Calculus I

Lecture 1

Introduction to functions and limits

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Introduction to lecture 1

This lecture will define key concepts and definition of terms that are commonly used calculus. This lecture will focus on these definitions that will form a foundation in comprehending other concepts and operations in other lectures in differential and integral calculus.

Intended learning outcomes

At the end of this lecture, you will be able to;

(i) Explain key terms relating to functions and limits of functions.

(ii) Determine limits of given functions.

References for further reading

The lecture notes have been adopted from relevant topics from (Kahenya, 2022; Stewart, 2012; Sullivan & Miranda, 2019)

Definition 1: A relation is a rule that relates a value in the domain with a value in the range. Functions are situation where one quantity depends on another. For example the area A of a circle depends on the radius r i.e. $A = \pi r^2$.

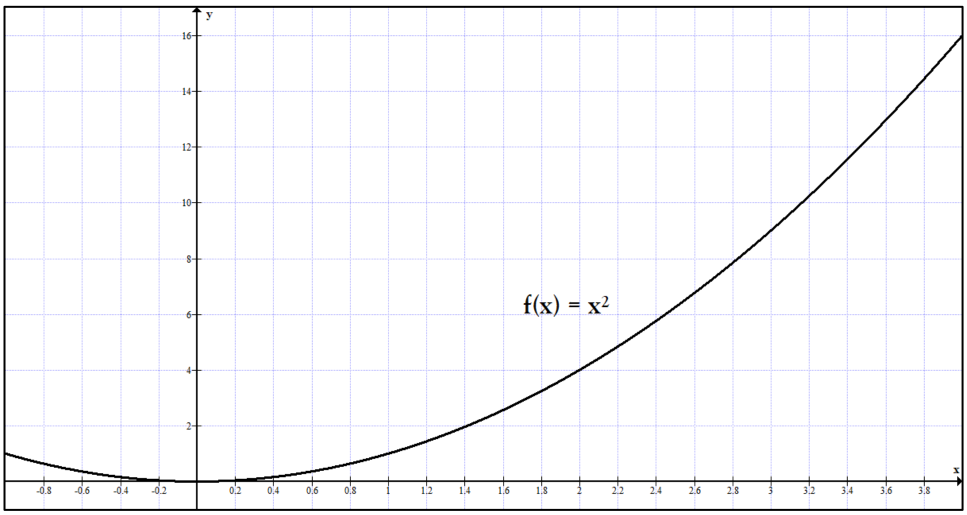
Definition 2: A function f is a rule that assigns to each element x in the Domain $\mathbb{X}$ exactly one element called $f(x)$ in a set $\mathbb{Y}$, called the Range. An arbitrary number in the domain of a function is called an independent variable, while a number in the range of the function is the dependent variable.

Example 1: In the function $A = \pi r^2$, A depends on r. Thus A is the dependent variable and r is the independent variable and therefore we can say that A is a function of r; written $A(r)$.

Definition 3: Domain is the set of all the values which are input in function. For example in the function $y = f(x)$, the domain is the set of all the x coordinates i.e. the independent variables.

Definition 4: Range is the set of all the values which are output of a function when inputs from the domain are used. The range constitutes the dependent variables. For example, in the function $y = f(x)$, the range is the set of all the y coordinates.

Example 1: Consider the graph of the function $y = f(x) = x^2$ from $x = 0$ to $x = 4$.



Note that the domain of the function is $0 \leq x \leq 4$ and the range is $0 \leq f(x) \leq 16$

Definition 5: A polynomial function is a function of the form;

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{n-1}x^{n-1} + a_nx^n$ where $n > 0$ i.e. n is a positive integer, and the coefficients $a_0, a_1, a_2, \ldots a_n$ are constants real numbers.

If $a_n \neq 0$, then the polynomial is of degree n e.g. $f(x) = 3x^4 + x^2 + 3x - 1$ is of degree 4 with coefficients $a_4 = 3, a_3 = 0, a_2 = 1, a_1 = 3, a_0 = -1$.

A polynomial function of the form $f(x) = b$ is called a constant function e.g. $f(x) = 3$.

A polynomial of the form $f(x) = a_1x + a_0$ is called a linear function.

Other functions that these course will delve in include algebraic functions, rational functions, discontinuous functions, trigonometric functions, logarithmic functions among others.

Definition 6: There are different intervals where a function can be defined.

a) **Bounded intervals**

i) **Open intervals:** Let a and b be fixed real numbers then the open interval (a, b) is the set of all real numbers x such that $\mathrm{a} < \mathrm{x} < \mathrm{b}$. Real numbers a and b are not part of the interval.

ii) **Closed intervals:** Let a and b are fixed real numbers then the closed interval $[\mathrm{a}, \mathrm{b}]$ is the set of all real numbers x such that $\mathrm{a} \leq \mathrm{x} \leq \mathrm{b}$. Real numbers a and b are part of the interval.

iii) **Half-open intervals:** These are intervals that are open on one side and closed on the other e.g. $[\mathrm{a}, \mathrm{b})$ or $(\mathrm{a}, \mathrm{b}]$.

b) **Unbounded intervals.** Example of unbounded intervals are;

i) (a, ∞) i.e. $x > a$

ii) $(-\infty, a)$ i.e.$x < a$

iii) $[a, \infty)$ i.e.$x \geq x$

iv) $(-\infty, a]$ i.e.$x \leq a$

Limits and their properties

Limit of a function concept is a fundamental concept in calculus.

Consider a spherical balloon. Suppose one has to pump air into the balloon such that it expands uniformly. If its radius is r , then its volume is $v = \frac{4}{3}\pi r^3$. Now, if we let the radius to approach or tends to 12 cm i.e. $r \to 12$, then its volume v approaches 2304π i.e. $\mathrm{v} \to 2304\pi$.

We can denote this as;

$$\lim_{r \to 12} \left(\frac{4}{3}\pi r^3\right) = 2304\pi$$

The idea of a function approaching a limit can be defined as the function $f(x)$ approaches the limit l near a point a if one can make $f(x)$ as close as to l as one would like, by having x sufficiently close to, but not equal to a. This can be denoted as;

$$\lim_{\mathrm{x} \to \mathrm{a}} \mathrm{f}(\mathrm{x}) = \mathrm{l}$$

Example 1: Consider the function $f(x) = x^2 + 1$ for the values of x near 2 but not equal to 2. The table below shows the values of $f(x)$ for the values of x in the neighborhood of 2. We investigate the values of $f(x)$ from either sides of 2. That is, when x is approaching 2 from the left and also from the right.

From the LHS

x	1	1.5	1.8	1.9	1.999	1.999999
$f(x)$	2	3.25	4.24	4.61	4.996001	4.999996000001

From the RHS

x	3	2.5	2.1	2.01	2.000001	2.0000001
$f(x)$	10	7.25	5.41	5.0401	5.000004000001	5.0000040000001

Remark 1: Note that for the limit of $f(x)$ to exist, it is NOT a MUST that $f(x)$ MUST be defined at point a.

Definition 1: The limit of $f(x)$ is l as long as the function $f(x)$ approaches to l as x tends to a from either side i.e.,

From the left; $\lim_{x \to a^-} f(x) = l$

From the right; $\lim_{x \to a^+} f(x) = l$

$$\Rightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = l$$

Remark 2: If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ then the limit of the function does not exist.

Theorem 1: If $\lim_{x\to a} f(x) = m$ and $\lim_{x\to a} g(x) = n$ then

a) $\lim_{x\to a}(f \pm g)(x) = m \pm n$

b) $\lim_{x\to a}(f \cdot g)(x) = m \cdot n$

c) $\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{m}{n}$

d) $\lim_{x\to a}[\alpha f(x)] = \alpha \lim_{x\to a} f(x) = \alpha m$, where α is a constant

Methods of finding limit of functions

a) **Limits by direct substitution**

Example 1: Evaluate; $\lim_{x\to 3}(3x + 5) = 3(3) + 5 = 14$

Example 2: Evaluate; $\lim_{X\to -2}(x^2 - 2) = (-2)^2 - 2 = 4 - 2 = 2$

b) **Factorization**

Direct substitution may be faulty in certain circumstances. Hence one need to first simplify the function or factorize it before substituting it accordingly.

Example 1: Evaluate; $\lim_{x\to 3}\left(\frac{5x^2-12x-9}{x-3}\right)$

Solution: Suppose we use direct substitution, we get;

$$\lim_{x\to 3}\left(\frac{5x^2 - 12x - 9}{x - 3}\right) = \frac{45 - 36 - 9}{3 - 3} = \frac{0}{0} - \text{indeterminate form}$$

The indeterminate form doesn't talk much about the limit of the functions. In such a situation one has to options; (i) factorize the numerator or (ii) use the L'Hopital's rule.

Factorizing the numerator to get; $5x^2 - 12x - 9 = (x - 3)(5x + 3)$ and hence we have;

$$\lim_{x\to 3}\left(\frac{5x^2 - 12x - 9}{x - 3}\right) = \lim_{x\to 3}\left(\frac{(x - 3)(5x + 3)}{x - 3}\right) = \lim_{x\to 3}(5x + 3) = 18$$

c) **L'Hopital's rule**

This rule is used where after direct substitution or otherwise, one gets the indeterminate forms;

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

Then the limit exists or not, and it's called an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Definition 1: (L'Hopital's rule) Let $f(x)$ and $g(x)$ be differentiable functions and $g'(x) \neq 0$ on the open interval I that contains a point x. Suppose;

$$\lim_{x\to a} f(x) = \pm\infty \; ; \; \lim_{x\to a} f(x) = 0; \; \lim_{x\to a} g(x) = 0;$$

Then;

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \text{ if the limit on the RHS exists}$$

Remarks

1) Limits not of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ may require another rule rather than the L'Hopital's rule.
2) Indeterminate forms of the type; $0 \cdot \infty; 0 \cdot -\infty; \; 0^0, \infty^0, and \; 1^\infty$ can be reduced to either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
3) For type $0^0, \infty^0, \text{and } 1^\infty$ apply logs of the expressions involved.

Example 1: Evaluate; $\lim_{x\to 0^+} x^a \ln x = 0 \cdot -\infty \; (a > 0) = \lim_{x\to 0^+} \frac{\ln x}{x^{-a}} = \frac{-\infty}{\infty}$

$$\Rightarrow \lim_{x\to 0^+} \frac{\frac{1}{x}}{-ax^{-a-1}} = \lim_{x\to 0^+} \frac{x^a}{-a} = 0$$

Example 2: Evaluate; $\lim_{x\to 0^+} x^x$

Solution: by direct substitution we get; $\lim_{x\to 0^+} x^x = 0^0 - \text{indeterminate form}$

Hence we let $y = x^x \Rightarrow \ln y = x \ln x$. Next we find the limits of both sides to get;

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0} x \ln x = 0$$

$$\therefore \lim_{x\to 0^+} x^x = \lim_{x\to 0^+} y = e^0 = 1$$

Example 3: Evaluate $\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x$

Solution: $\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x = 1^\infty$. Let $y = \left(1 + \frac{2}{x}\right)^x \Rightarrow \ln y = x \ln\left(1 + \frac{2}{x}\right)$

Thus; $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(1 + \frac{2}{x}\right) = \infty \cdot 0 \Rightarrow = \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$

Applying L'Hopital's rule to get;

$$\lim_{x\to\infty} \frac{\ln\left(1 + \frac{2}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{-\frac{2}{x(x+2)}}{-\frac{1}{x^2}} = 2 \therefore \lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x = e^2$$

d) **Rationalizing the function**

Example 1: Evaluate the following; $\lim_{x \to 4}(4 - \sqrt{x})$

Solution: $\lim_{x \to 4}(4 - \sqrt{x}) = 2$

Example 2: Evaluate the following; $\lim_{x \to \infty}\left(\frac{\sqrt{x}-5}{x-25}\right)$

Solution: $\lim_{x \to \infty}\left(\frac{\sqrt{x}-5}{x-25}\right) = \lim_{x \to \infty}\frac{\sqrt{x}-5}{(\sqrt{x}-5)(\sqrt{x}+5)} = \lim_{x \to \infty}\frac{1}{\sqrt{x}+5} = 0$

Example 3: Evaluate the following; $\lim_{x \to 0}\left(\frac{\sqrt{x+9}-3}{x}\right)$

Solution: Direct substitute we get $\frac{0}{0}$ - indeterminate.

$$\lim_{x \to 0}\left(\frac{\sqrt{x+9}-3}{x}\right) = \lim_{x \to 0}\left(\frac{\sqrt{x+9}-3}{x} \times \frac{\sqrt{x+9}+3}{\sqrt{x+9}+3}\right)$$

$$= \lim_{x \to 0}\frac{x+9-9}{x(\sqrt{x+9}+3)}$$

$$= \lim_{x \to 0}\frac{1}{(\sqrt{x+9}+3)} = \frac{1}{6}$$

e) **Limits at infinity**

Suppose the function $f(x)$ gets arbitrarily close to l as x becomes sufficiently large, then we say that $f(x)$ has a limit at infinity denoted; $\lim_{x \to \infty} f(x) = l$

Similarly; Suppose the function $f(x)$ gets arbitrarily close to l for $x < 0$ as $|x|$ becomes sufficiently large, then we say that $f(x)$ has a limit at negative infinity denoted; $\lim_{x \to -\infty} f(x) = l$.

Note that if the function gets arbitrarily close to las $x \to \pm\infty$, the graph of $f(x)$ approaches the line $f(x) = l$ which is a horizontal asymptote of $f(x)$.

f) **Infinite limits at infinity**

A function $f(x)$ is said to have a positive (negative) infinite limit at infinity, as $f(x)$ (or $f(x) < 0$ and $|f(x)|$) becomes arbitrarily large for x sufficiently large denoted; $\lim_{x \to \pm\infty} f(x) = \pm\infty$

Example 1: Evaluate the following: $\lim_{x \to \infty}\left(\frac{2x^2+7}{3x^3-8}\right)$

Solution: By direct substitution we get $\frac{\infty}{\infty}$ - indeterminate

We need to divide every term by the highest power of the variable in the denominator i.e. x^3 to get;

$$\lim_{x\to\infty}\left(\frac{2x^2+7}{3x^3-8}\right) = \lim_{x\to\infty}\left(\frac{\frac{2x^2}{x^3}+\frac{7}{x^3}}{\frac{3x^3}{x^3}-\frac{8}{x^3}}\right) = \lim_{x\to\infty}\left(\frac{\frac{2}{x}+\frac{7}{x^3}}{3-\frac{8}{x^3}}\right) = \frac{0+0}{3-0} = 0$$

Example 2: Evaluate the following: $\lim_{x\to\infty}\left(\frac{2x^4+11}{3x^3+x+1}\right)$

Solution: By direct substitution we get $\frac{\infty}{\infty}$ - indeterminate

We need to divide every term by the highest power of the variable in the denominator i.e. x^3 to get;

$$\lim_{x\to\infty}\left(\frac{2x^4+11}{3x^3+x+1}\right) = \lim_{x\to\infty}\left(\frac{\frac{2x^4}{x^3}+\frac{11}{x^3}}{\frac{3x^3}{x^3}+\frac{x}{x^3}+\frac{1}{x^3}}\right) = \lim_{x\to\infty}\left(\frac{2x+\frac{11}{x^3}}{3+\frac{1}{x^2}+\frac{1}{x^3}}\right) = \infty$$

Example 3: Evaluate the following: $\lim_{x\to\infty}\left(\frac{x^4-1}{3x^4+x^2+1}\right)$

Solution: By direct substitution we get $\frac{\infty}{\infty}$ - indeterminate

We need to divide every term by the highest power of the variable in the denominator i.e. x^4 to get;

$$\lim_{x\to\infty}\left(\frac{\frac{x^4}{x^4}-\frac{1}{x^4}}{\frac{3x^4}{x^4}+\frac{x^2}{x^4}+\frac{1}{x^4}}\right) = \lim_{x\to\infty}\left(\frac{1-\frac{1}{x^4}}{3+\frac{1}{x^2}+\frac{1}{x^4}}\right) = \frac{1}{3}$$

Exercise

1) Define the following terms as used in calculus and give an example to support your response (you can also use illustrations); a limit of a function, and Intervals of a function
2) Explain how the concept of limit is applicable in real-life scenario. (its relevance to the physical world) Illustrate with a clear example.
3) Evaluate the following;
   a) $\lim_{x\to -2}(2x^3+x-7)$
   b) $\lim_{x\to 3}\left(\frac{1}{3x+2}\right)$
   c) $\lim_{x\to 0}\frac{(3x-7)}{x}$
   d) $\lim_{x\to 5}\frac{2x^2-17x+35}{x-5}$
   e) $\lim_{x\to\infty}(3x^2+4)$
   f) $\lim_{x\to\infty}\frac{3x^2+2x+1}{7x^3-x+2}$

References

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