Calculus I

Lecture 2

Limits of functions

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Introduction to lecture 2

This lecture is a continuation of lecture 1. The lecture discusses limits of functions.

Intended learning outcomes

At the end of this lecture, you will be able to;

(i) Explain key terms in functions and limits of functions.

(ii) Carry out operations involving limits of function

References for further reading

The lecture notes have been adopted from relevant topics from (Kahenya, 2022; Stewart, 2012; Sullivan & Miranda, 2019).

Limits of piecewise functions

Example 1: Given a piecewise function $f(x)$;

$$f(x) = \begin{cases} g(x), & \text{if } x > a \\ h(x), & \text{if } x \leq a \end{cases}$$

then the limit l of the function $f(x)$ as x approaches a exists if and only if;

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$

That is, we need to find the one-sided limits of $f(x)$ as x approaches a;

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x)$$

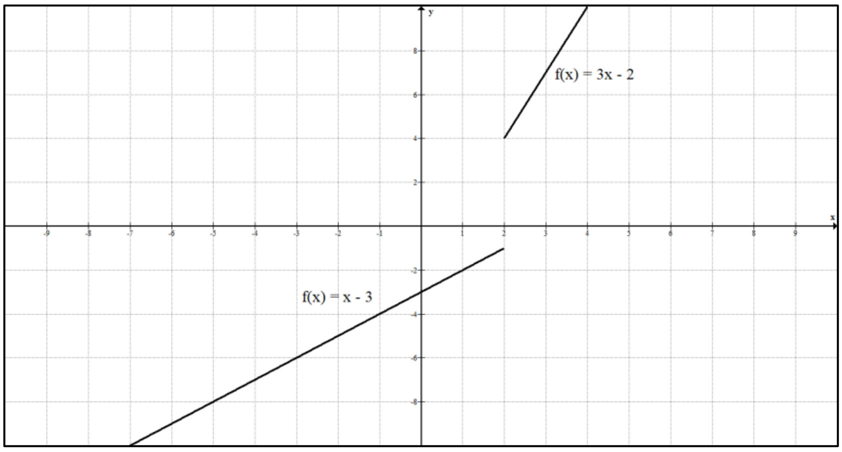
Example 2: Determine if the $\lim_{x \to 2} f(x)$ exists for the function;

$$f(x) = \begin{cases} 3x - 2, & \text{if } x > 2 \\ x - 3, & \text{if } x < 2 \end{cases}$$

Solution: We need to show if $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$, if the limit indeed exists.

Hence; $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x - 2) = 4$; $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x - 3) = -1$

Clearly; $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$. Therefore, the limit of the function doesn't exist (see figure below).



Definition 1 (the $\boldsymbol{\varepsilon - \delta}$ definition of a limit of a function)

In lecture 1 we had some definition of the limit of a function. For instance, given the function $\mathrm{f(x)}$ then x is near say a point a but not equal to a and we can introduce a δ that represents a small positive change in x in such a way we can say that the value x satifies;

$$|\mathrm{x} - \mathrm{a}| < \delta$$

To imply the interval $\mathrm{a} - \delta < \mathrm{x} < \mathrm{a} + \delta$.

Similarly, since as x tends to a,the function $\mathrm{f(x)}$ approaches a limit l, then we can also introduce a ε to represent a positive change in $\mathrm{f(x)}$ that is;

$$|\mathrm{f(x)} - \mathrm{l}| < \varepsilon$$

To mean that $\mathrm{f(x)}$ is near the limit l. We can conclude the above as follows:

Let $f(x)$ be a function defined everywhere in an open interval containing point a, except possibly at point a. Then the limit of $f(x)$ as x approaches point a is l denoted;

$$\lim_{x \to a} f(x) = l$$

If given any number, $\varepsilon > 0$ there exists a number $\delta > 0$ such that whenever $0 < |x - a| < \delta$ then $|f(x) - l| < \varepsilon$.

Example 1: Use the $\varepsilon - \delta$ definition of a limit to show that the limit of $f(x) = 3x - 2$ is 4 as x approaches 2 i.e. $\lim_{x \to 2}(3x - 2) = 4$

Solution: by definition, given $\varepsilon > 0$ there exists a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - l| < \varepsilon$.

In this case; $f(x) = 3x - 2, l = 4, a = 2$

Our task is, for all positive ε, we need to find $\delta > 0$ such that;

$$0 < |x - 2| < \delta \Rightarrow |(3x - 2) - 4| < \varepsilon$$

Next, we consider the RHS which is equivalent to;

$$|(3x - 2) - 4| < \varepsilon \sim |3x - 2 - 4| < \varepsilon$$

$$\sim |3x - 6| < \varepsilon$$

$$\sim 3|x - 2| < \varepsilon$$

$$\sim |x - 2| < \frac{\varepsilon}{3}$$

We can conclude that if $|x - 2| < \frac{\varepsilon}{3}$ then $|(3x - 2) - 4| < \varepsilon$ and therefore we choose a $\delta = \frac{\varepsilon}{3}$.

Hence 4 is the limit.

Example 2: use the $\varepsilon - \delta$ definition of a limit to show that the limit of $f(x) = x^2 + 8x + 15$ is 48 as x approaches 3 i.e. $\lim_{x \to 3}(x^2 + 8x + 15) = 48$

Solution: : by definition, given $\varepsilon > 0$ there exists a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - l| < \varepsilon$.

In this case; $f(x) = x^2 + 8x + 15, l = 48, a = 3$

Our task is, for all positive ε, we need to find $\delta > 0$ such that;

$$0 < |x - 3| < \delta \Rightarrow |(x^2 + 8x + 15) - 48| < \varepsilon$$

The RHS is equivalent to;

$$\sim |x^2 + 8x + 15 - 48| < \varepsilon$$

$$\sim |x^2 + 8x - 33| < \varepsilon$$

$$\sim |(x - 3)(x + 11)| < \varepsilon$$

$$\sim |x - 3||x + 11| < \varepsilon$$

Hence, we have; $|x - 3| < \frac{\varepsilon}{|x+11|}$

We let $\delta = \frac{\varepsilon}{|x+11|}$

Since from the definition x must be close to point a, we can restrict our x such that it is at most 1 unit away from the point $a = 3$ i.e. $|x - 3| < 1 \cdots$ (i)

Therefore, in our case;

$$|x-3| < 1 \Rightarrow -1 < x-3 < 1 \therefore 2 < x < 4$$

therefore $|x+11|$ will then be in the range $13 < x+11 < 15$

For the inequality $|x-3| < \frac{\varepsilon}{x+11}$ the RHS will be at its minimum when $x+11$ is at its maximum (i.e. at 15). Hence;

$$|x-3| < \frac{\varepsilon}{x+11} < \frac{\varepsilon}{15} \cdots \text{(ii)}$$

If we consider inequalities (i) and (ii) we have two restrictions i.e.

$$|x-3 < 1 \text{ and } |x-3| < \frac{\varepsilon}{15}$$

We choose $\delta = \min\left\{1, \frac{\varepsilon}{15}\right\}$ i.e., we take the smaller of these two values. Indeed 48 is the limit.

Example 3: Apply the $\varepsilon - \delta$ definition to prove that; $\lim\limits_{x\to 4} x^2 = 16$

Proof: by definition, given $\varepsilon > 0$ there exists a $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x) - l| < \varepsilon$.

In this case; $f(x) = x^2, l = 16, a = 4$

Our task is, for all positive ε, we need to find $\delta > 0$ such that;

$$0 < |x-4| < \delta \Rightarrow |x^2 - 16| < \varepsilon$$

Next, we consider the RHS which is equivalent to;

$$|x^2 - 16| < \varepsilon \sim |(x-4)(x+4)| < \varepsilon$$

$$\sim |x-4||x+4| < \varepsilon$$

$$\sim |x-4| < \frac{\varepsilon}{|x+4|}$$

Hence, we have; $|x-4| < \frac{\varepsilon}{|x+4|}$

We let $\delta = \frac{\varepsilon}{|x+4|}$

Since from the definition x must be close to point a, we can restrict our x such that it is at most 1 unit away from the point $a = 4$ i.e. $|x-4| < 1 \cdots (i)$

Therefore, in our case;

$$|x-4| < 1 \Rightarrow -1 < x-4 < 1 \therefore 3 < x < 5$$

Therefore $|x+4|$ will then be in the range $7 < x+4 < 9$

For the inequality $|x-4| < \frac{\varepsilon}{x+4}$ the RHS will be at its minimum when $x+4$ is at its maximum (i.e. at 9).

Hence;

$$|\mathrm{x}-4| < \frac{\varepsilon}{\mathrm{x}+4} < \frac{\varepsilon}{9} \cdots \text{(ii)}$$

If we consider inequalities (i) and (ii) we have two restrictions i.e.

$$|\mathrm{x}-4 < 1 \text{ and } |\mathrm{x}+4| < \frac{\varepsilon}{9}$$

We choose $\delta = \min\left\{1, \frac{\varepsilon}{9}\right\}$ i.e. we take the smaller of these two values.

Example 4: Show that $\lim_{\mathrm{x}\to 1}(\mathrm{x}^3+2\mathrm{x}+3) = 6$

Proof: by definition, given $\varepsilon > 0$ there exists a $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x) - l| < \varepsilon$.

In this case; $\mathrm{f(x)} = \mathrm{x}^3 + 2\mathrm{x} + 3, l = 6, a = 1$

Our task is, for all positive ε, we need to find $\delta > 0$ such that;

$$0 < |x-1| < \delta \Rightarrow |\ (\mathrm{x}^3 + 2\mathrm{x} + 3) - 6| < \varepsilon$$

Working with the RHS:

$$|\ (\mathrm{x}^3 + 2\mathrm{x} + 3) - 6| < \varepsilon \sim |\ \mathrm{x}^3 + 2\mathrm{x} - 3| < \varepsilon$$

$$\sim |(\mathrm{x}^2 + \mathrm{x} + 3)(\mathrm{x}-1)| < \varepsilon$$

$$\sim |\mathrm{x}-1||\mathrm{x}^2 + \mathrm{x} + 3| < \varepsilon$$

$$\sim |\mathrm{x}-1| < \frac{\varepsilon}{\mathrm{x}^2 + \mathrm{x} + 3}$$

We let $\delta = \frac{\varepsilon}{\mathrm{x}^2+\mathrm{x}+3}$

Since from the definition x must be close to point a, we can restrict our x such that it is at most 1 unit away from the point $a = 1$ i.e. $|x-1| < 1 \cdots (i)$

Therefore, in our case;

$$|x-1| < 1 \Rightarrow -1 < x-1 < 1 \therefore 0 < x < 2$$

We can have that $|x| < 2$

Therefore $\mathrm{x}^2 + \mathrm{x} + 3 \leq |x||x| + |x| + 3 < 4 + 2 + 3 = 9$

Hence $|\mathrm{x}-1| < \frac{\varepsilon}{\mathrm{x}^2+\mathrm{x}+3} < \frac{\varepsilon}{9} \cdots \text{(ii)}$

We can then choose a $\delta = \min\left\{1, \frac{\varepsilon}{9}\right\}$

Definition 2: (Infinite limit). Suppose $f(x)$ is defined everywhere on an open interval containing point a. Then $f(x)$ becomes unbounded in the positive direction or has an infinite limit i.e. $\lim_{x\to a} f(x) = \infty$

If for every positive number M, there exists a positive delta i.e. $\delta > 0$ such that whenever

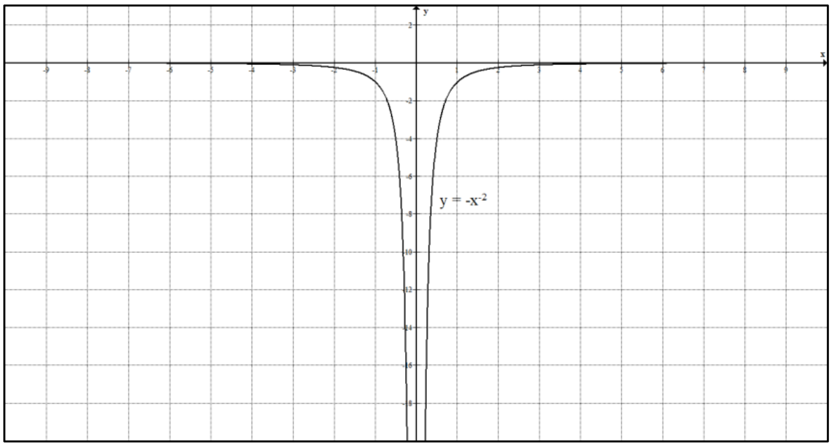
$0 < |x - a| < \delta$ then $f(x) > M$.

Similarly, a function $f(x)$ is unbounded in the negative direction i.e. $\lim_{x\to a} f(x) = -\infty$

If, for every negative number N i.e. $N < 0$ there exists $\delta > 0$ such that whenever;

$0 < |x - a| < \delta$ then $f(x) < N$.

Example 1: $\lim_{x\to 0}\left(\frac{1}{x^2}\right) = \infty$ or $\lim_{x\to 0}\left(-\frac{1}{x^2}\right) = -\infty$ (see figure below)

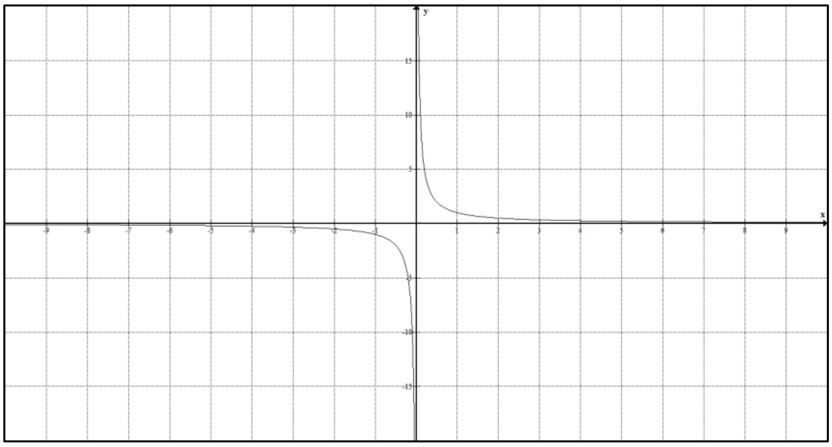


Definition 3: Vertical asymptote

The line $x = a$ is called a vertical asymptote of the graph of the function $f(x)$ if

$$\lim_{x\to a} f(x) = \pm\infty$$

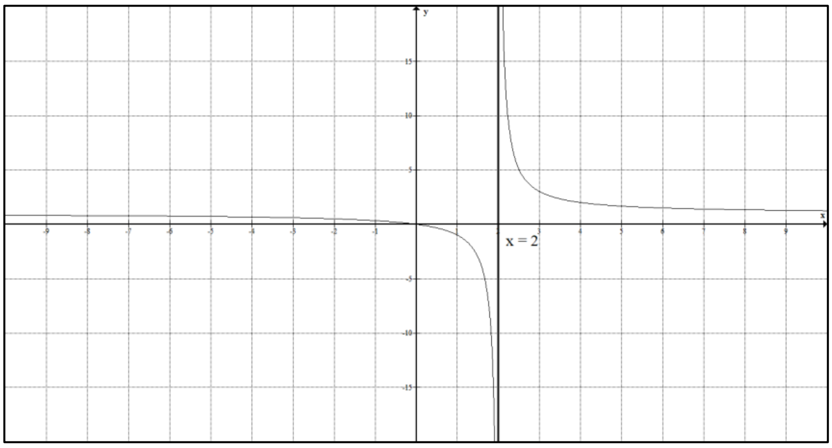
Example 1: Given that the $\lim_{x\to 0}\left(\frac{1}{x}\right) = \pm\,\infty$ then the vertical asymptote is the line $x = 0$ (see figure below).



Example 2: Evaluate $\lim\limits_{x\to 2}\left(\frac{x}{x-2}\right)$ and determine the vertical asymptote.

Solution: $\lim\limits_{x\to 2^+}\left(\frac{x}{x-2}\right)=\infty$ also $\lim\limits_{x\to 2^-}\left(\frac{x}{x-2}\right)=-\infty$

The vertical asymptote is the line $x=2$ (see the figure below).



Exercise

1) Determine if the limits of the following piece-wise functions exist.

a) $f(x)=\begin{cases}3x+1 \text{ if } x>4\\ x^2-1 \text{ if } x\le 4\end{cases}$

b) $g(x)=\begin{cases}2-x \text{ if } x<0\\ 6x+2 \text{ if } x\ge 0\end{cases}$

c) $h(x)=\begin{cases}\frac{x^2-1}{x-1} \text{ if } x\neq 1\\ 2 \text{ if } x=1\end{cases}$ at 2

d)

2) Use the $\varepsilon-\delta$ definition to show if the limits of the following functions exist.

a) $\lim\limits_{x\to -2}(4x+7)=-1$

b) $\lim\limits_{x\to 3}\left(\frac{5x^2-8x-21}{x-3}\right)=23$

c) $\lim\limits_{x\to 1}(x^2+3x+2)=6$

d) $\lim\limits_{x\to -5}(2x^2-11x-21)=84$

e) $\lim\limits_{x\to \infty}3x^2=\infty$

f) $\lim\limits_{x\to 1}(4x^2+7x+3)=14$

g) $\lim\limits_{x\to 2}(x^3-2x+1)=5$

h) $\lim\limits_{x\to -3}(x^3+2x+1)=-32$

References

Kahenya, N. P. (2022). *Mathematics for science*. https://www.hufocw.org/Course/360

Stewart, J. (2012). *Calculus* (7th ed.). BROOKS/COLE Cengage Learning.

Sullivan, M., & Miranda, K. (2019). *Calculus: Early Transcendentals* (second). W.H. Freeman and Company.