Calculus I

Lecture 3

Continuity of functions

Lecturer: Kahenya, N.P

Introduction to lecture 3

This lecture is a continuation of lectures 1 and 2. It will focus on the definitions of continuous functions that form a foundation in comprehending other concepts and operations in other lectures in differential and integral calculus.

Intended learning outcomes

At the end of this lecture, you will be able to;

(i) Explain key terms in continuous functions.

(ii) Carry out operations involving these key concepts.

References for further reading

The lecture notes have been adopted from relevant topics from (Kahenya, 2022; Stewart, 2012; Sullivan & Miranda, 2019).

Definition 1: A function f is continuous if its graph has no breaks. That is, one can draw its graph without lifting the pen.

Definition 2: A function f is continuous on the interval I if it is continuous at every point of this interval I.

Definition 3: A function f is continuous at the point a if its limit as it approaches a is the same as $f(a)$ i.e.

$$\lim_{x \to a} f(x) = f(a)$$

Example 1: $\lim_{x\to 3}(x^2+1)=10$. Note that $f(3)=10 \Rightarrow \lim_{x\to a} f(x)=f(a)$

Definition 4: A function f is said to be right continuous at point a if;

$$\lim_{x\to a^+} f(x)=f(a)$$

A function f is said to be left continuous at point a if;

$$\lim_{x\to a^-} f(x)=f(a)$$

Definition 5: A function f is continuous at a point x_0 if given $\varepsilon>0, \exists\, \delta>0$ such that:

$|f(x)-f(x_0)|<\varepsilon$ provided $|x-x_0|<\delta$

Definition 6: A function $f(x)$ is said to be continuous at an open interval I if and only if it is continuous at every point on the interval i.e., given $\varepsilon>0$, $\exists\, \delta>0$ such that $|f(x)-f(y)|<\varepsilon$, provided for every $y\in I$ implies $|x-y|<\delta$.

Example 1: Show that $f(x)=2x$ is continuous on $\mathbb{R}$.

Solution: By definition $\forall \varepsilon>0, \exists \delta>0$ such that $|f(x)-f(y)|<\varepsilon$ whenever $|x-y|<\delta$ and $x,y\in\mathbb{R}$.

$$|2x-2y|=2|x-y|<\varepsilon \Rightarrow |x-y|<\frac{\varepsilon}{2}$$

We choose $\delta=\frac{\varepsilon}{2}$. This δ does not depend on the choice of y. This implies that the function is uniformly continuous on $\mathbb{R}$.

Definition 7: A function $f(x)$ is said to be continuous at a point x_0 if the following holds;

i) $f(x_0)$ is defined.

ii) $\lim_{x\to x_0} f(x)$ exists.

iii) $\lim_{x\to x_0} f(x)=f(x_0)$

Properties of continuous functions

Theorem 1: Basic algebraic properties of continuous functions

(i) If f and g are continuous at point x_0, then so are $f+g, f-g, f.g$

(ii) If f and g are continuous at point x_0, and $g(x)\neq 0$, then $\frac{f}{g}$ is continuous at x_0

(iii) A polynomial function is continuous at every point.

(iv) A rational function is continuous at every number for which it is defined.

(v) Sine and cosine functions are continuous at every point.

(vi) Tangent, cotangent, secant, and cosecant functions are continuous at every point for which they are defined.

(vii) If g is continuous at x_0 and f is continuous at $g(x_0)$, then the function h, defined by $h(x) = f[g(x)]$ is continuous at x_0

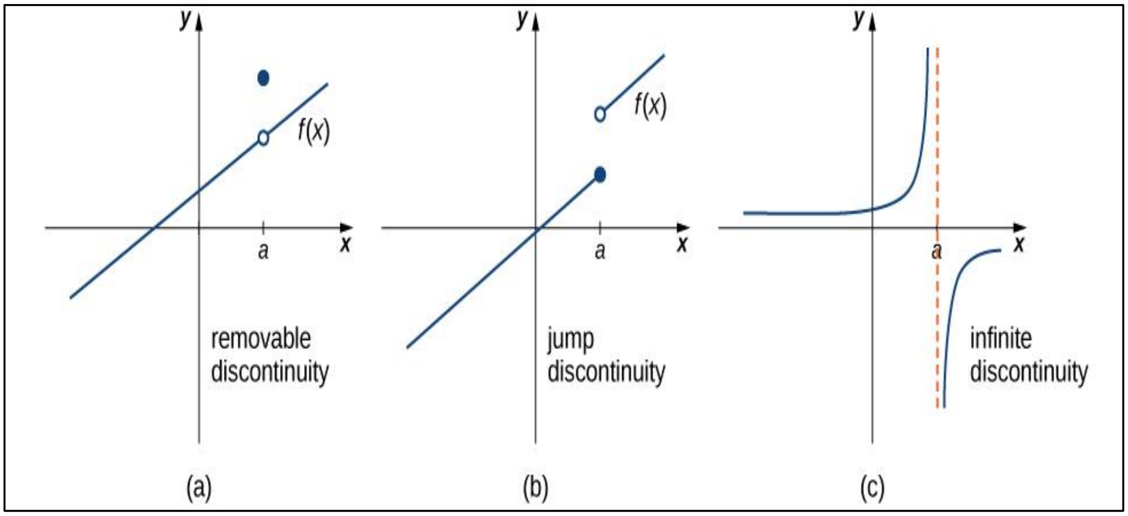
Definition 1: Types of discontinuity

We can categorize discontinuities into three categories (see diagram below);

(i) Removable discontinuity where there is a hole in the graph of the function. If a function $f(x)$ has a removable discontinuity say at point x_0 then its limit exists i.e. $\lim_{x \to x_0} f(x) = l$ where $l \in \mathbb{R}$.

(ii) Jump discontinuity is where the limits of the function from both sides of a point x_0 are not the same but are real numbers. That is $\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x)$.

(iii) Infinite discontinuity is where a function $f(x)$ at a point x_0 has it limits as $\lim_{x \to x_0-} f(x) = \pm\infty$ or $\lim_{x \to x_0^+} f(x) = \pm\infty$



Source: Strang & Herman, (2022).

Example 1: Determine if the function $f(x) = \begin{cases} \frac{2x^2+3x+1}{x+1} & \text{if } x \neq -1 \\ 3 & \text{if } x = -1 \end{cases}$ is continuous at -1.

To determine this, we use definition 7 above i.e. A function $f(x)$ is said to be continuous at a point x_0 if the following holds;

i) $f(x_0)$ is defined

ii) $\lim_{x \to x_0} f(x)$ exists

iii) $\lim_{x \to x_0} f(x) = f(x_0)$

Now;

(i) For $x_0 = -1; f(-1) = 3 \Rightarrow f(-1)$ is defined.

(ii) For $x_0 \neq -1, f(x) = \frac{[(2x+1)(x+1)]}{x+1} = 2x + 1$

$$\Rightarrow \lim_{x \to -1} f(x) = \lim_{x \to -1} (2x + 1) = -1$$

Therefore $\lim_{x \to -1} f(x)$ exists

But $\lim_{x \to -1} f(x) = -1$ and $f(-1) = 3 => \lim_{x \to x_0} f(x) \neq f(x_0)$

Thus, the function is discontinuous at -1.

Example 2: Let $g(x) = \begin{cases} 3 - x^2 & \text{if } x \leq 1 \\ 1 + x^2 & \text{if } x > 1 \end{cases}$ Determine if $g(x)$ is continuous at $x = 1$.

Solution: Here we use definition 4.

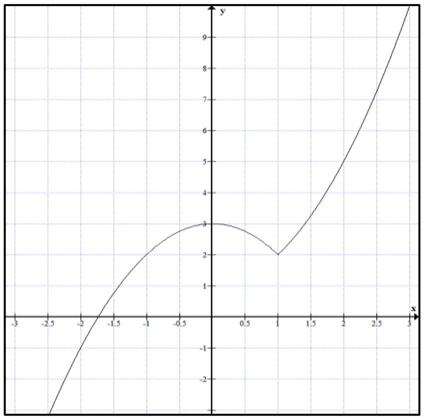
Now; $\lim_{x \to 1^-} (3 - x^2) = 2$

$\lim_{x \to 1^+} (1 + x^2) = 2$. This implies the limit $\lim_{x \to 1^+} g(x) = 2$.

Again, note that at; $x = 1, g(1) = 2$

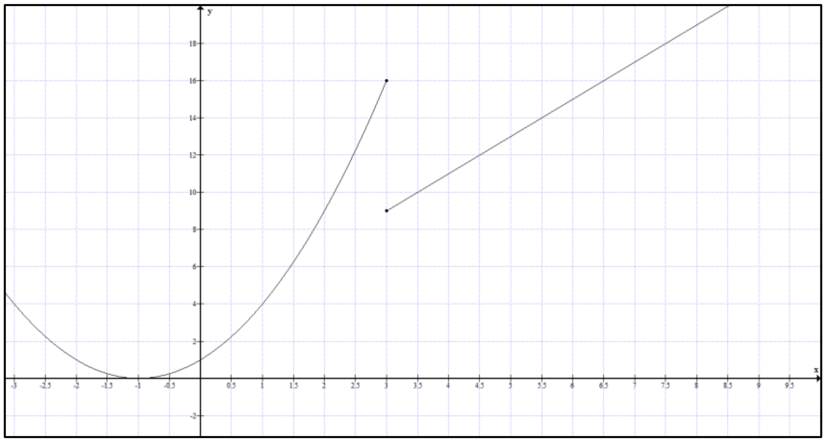
From definition 3 (iii) we can see that; $\lim_{x \to 1} g(x) = g(x) = 2$

Hence g is continuous at $x = 1$ (See figure on the right).



Example 3: Determine if the function $\mathrm{f(x)} = \begin{cases} \mathrm{x^2 + 2x + 1 \text{ if } x < 3} \\ \mathrm{2x + 3 \text{ if } x \geq 3} \end{cases}$ is continuous at point $\mathrm{x = 3}$

Solution: We need to show if $\lim_{x \to 3^-} \mathrm{f(x)} = \lim_{x \to 3^+} \mathrm{f(x)}$



Thus;

$\lim_{x \to 3^-} (\mathrm{x^2 + 2x + 1}) = 9 + 6 + 1 = 16$

$\lim_{x \to 3^+} (\mathrm{2x + 3}) = 6 + 3 = 9$

$\Rightarrow \lim_{x \to 3^-} \mathrm{f(x)} = 16 \neq 9 = \lim_{x \to 3^+} \mathrm{f(x)}$

Hence the function is discontinuous at $x = 3$ i.e., jump discontinuity. (See figure on the right).

Example 4: Show that the function $\mathrm{f(x)} = 3 - \sqrt{9 - \mathrm{x}^2}$ is continuous on the [-3, 3].

Solution: We consider a point $a \in [-3,3]$ where $-3 < a < 3$ and proceed to determine the limit of the function as x approaches point a i.e.

$$\lim_{x \to a} \mathrm{f(x)} = \lim_{x \to a} \left(3 - \sqrt{9 - \mathrm{x}^2}\right) = 3 - \lim_{x \to a} \sqrt{9 - \mathrm{x}^2} = 3 - \sqrt{9 - \mathrm{a}^2} = \mathrm{f(a)}$$

This means that $\lim_{x \to a} \mathrm{f(x)} = \mathrm{f(a)}$ and therefore by definition of a limit the function is continuous on the interval $[-3,3]$

Example 5: Determine the value of k such that $f(x)$ is a continuous function.

$$\mathrm{f(x)} = \begin{cases} \mathrm{x^3 + 1 \text{ for } x < 1.5} \\ \mathrm{k(x - 6) \text{for } x \geq 1.5} \end{cases}$$

Solution: For the function to be continuous at point $x = 1.5$ then

$$\lim_{x \to 1.5^-} \mathrm{f(x)} = \lim_{x \to 1.5^+} \mathrm{f(x)}$$

Thus, we have; $\lim_{x \to 1.5^-} (x^3 + 1) = 4.375$

Therefore;

$$\lim_{x \to 1.5^+} \mathrm{k(x - 6)} = 4.375$$

$$\Rightarrow \mathrm{k(1.5 - 6)} = 4.375$$

$$\mathrm{k} = -\frac{4.375}{4.5} = -\frac{35}{36}$$

Example 6: Determine at what is the following function discontinuous

$$\mathrm{f(x)} = \frac{3\mathrm{x}^2 + 4\mathrm{x} + 1}{\mathrm{x} - 3}$$

Solution: It is discontinuous at $\mathrm{x} = 3$ i.e., a removable discontinuity, since we shall have

$$\mathrm{f(x)} = \frac{40}{0} = \infty$$

Example 7: Determine at what point(s) the following function is discontinuous.

$$\mathrm{f(x)} = \frac{\mathrm{x}^2 + \mathrm{x} + 1}{\mathrm{x}^3 + 1}$$

Solution: For the function to be discontinuous the denominator needs to be zero i.e.

$$\mathrm{x}^3 + 1 = 0 \Rightarrow \mathrm{x} = -1$$

Example 8: Determine if the function $y = \cos{(x^2 + 1)}$ is discontinuous.

Solution:

Note that we have a composite function $\mathrm{y} = \mathrm{f(g(x))}$ where $\mathrm{g(x)} = (\mathrm{x}^2 + 1)$ and $\mathrm{f(x)} = \cos \mathrm{x}$ It is clear that the function $g(x) = x^2 + 1$ is continuous on ℝ since it is a polynomial and function $f(x) = \cos x$ is continuous everywhere and hence the composite function y is continuous on ℝ

Theorem: The Intermediate value theorem

Suppose f is a continuous function on a closed interval [a, b] and $\mathrm{f(a)} \neq \mathrm{f(b)}$. Let N be any number between $\mathrm{f(a)}$ and $\mathrm{f(b)}$ i.e. $\mathrm{f(a)} < \mathrm{N} < \mathrm{f(b)}$ then there exist a number $\mathrm{x_o} \in (\mathrm{a}, \mathrm{b})$ such that $\mathrm{f(x_o)} = \mathrm{N}$.

Remark 1: This theorem is used to determine the zeros of a function.

Example 1: Apply the Intermediate value theorem to show that $\mathrm{f(x)} = \mathrm{x}^2 + 2\mathrm{x} - 3$ has a zero in the interval $(-2,3)$.

Solution: The function is a polynomial and hence it is continuous on the closed interval [-2, 3]. By IVT there exists a $\mathrm{x_0} \in (-2,3)$ such that $\mathrm{f(x_0)} = 0$

$$\Rightarrow \mathrm{f(x_0)} = \mathrm{x_0^2} + 2\mathrm{x_0} - 3 = 0$$

$$\Rightarrow (\mathrm{x_0} - 1)(\mathrm{x_0} + 3) = 0 \therefore \mathrm{x_0} = 1$$

Example 2: Apply the Intermediate Value Theorem to show that $f(x) = x^3 - 4x$ has a zero in the interval $[-3,3]$.

Solution: The function is a polynomial and hence it is defined on the closed interval. By IVT there exists a $x_o \in [-3,3]$ such that $\mathrm{f}(\mathrm{x}_0) = 0$

$$\Rightarrow \mathrm{f}(\mathrm{x}_0) = \mathrm{x}_0^3 - 4\mathrm{x}_0 = 0 \Rightarrow \mathrm{x}_0(\mathrm{x}_0^2 - 4) = 0$$

$$\therefore \mathrm{x}_0 = 0,$$

$$\text{Also } \mathrm{x}_0^2 - 4 = 0 \Rightarrow \mathrm{x}_0 = \pm 2$$

Exercise

1) Determine whether each of the following function sis continuous or discontinuous on each of the interval

   a) $f(x) = \sqrt{5 - x^2}$ $[-5,5], [6,10], (-3,2), (-1,5)$

   b) $g(x) = \frac{1}{x+1}; (-\infty, 1), (-5, -1), (-1, \infty), [-1, \infty)$

   c) $f(x) = \frac{x+3}{x^2-9}; (-\infty, 3], (-\infty, 3], (-\infty, -3], (-3, \infty), [-3,3]$

   d) $f(c) = \csc x; \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], (0, \pi), [0, \pi]$

2) Determine if $f(x) = \begin{cases} x - 2 \text{ if } x < 1 \\ 5 - 3x \text{ if } x \geq 1 \end{cases}$ is continuous at $x = 1$

3) Determine if the following functions are continuous at the given points

   a) $f(x) = \begin{cases} \frac{x^2-16}{x-4} \text{ if } x \neq 4 \\ 8 \text{ if } x = 4 \end{cases}$ at 4

   b) $g(x) = \begin{cases} \frac{x^2-1}{x-1} \text{ if } x \neq 1 \\ 2 \text{ if } x = 1 \end{cases}$ at 2

4) The cost of transporting goods between two points is given by the equation.

$$C(x) = \begin{cases} 0.5x \text{ if } 0 < x \leq 100 \\ 0.4x \text{ if } 100 \leq x < 200 \\ 0.2x \text{ if } 200 \leq x \end{cases}$$

   where x is the number of kgs being transported and $C(x)$ is the charge in KES. Sketch the graph of C and find these values of x for which C is discontinuous.

5) Apply the Intermediate Value Theorem to determine the zeros of the following functions in the given domain.

   a) $f(x) = 3x^2 + 8x + 2, (-3, 0)$

   b) $f(x) = x^2 + 2x + 1, (-2,1)$

6) Determine at what point are the following functions discontinuous or otherwise.

   a) $f(x) = \frac{3}{x}$

   b) $f(x) = \tan(\sin x)$

   c) $f(x) = \frac{(x+5)(3x-7)}{x+5}$

   d) $f(x) = \frac{x^3+2}{x^2-4}$

References

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