**Calculus I**

**Lecture 4**

**First Principle of Differentiation**

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**Introduction to lecture 4**

This lecture is a continuous of lectures 1,2, and 3. It will show the relationship between limits and differentiation.

**Intended learning outcomes**

At the end of this lecture, you will be able to;

(i) Explain the first principle of differentiation.

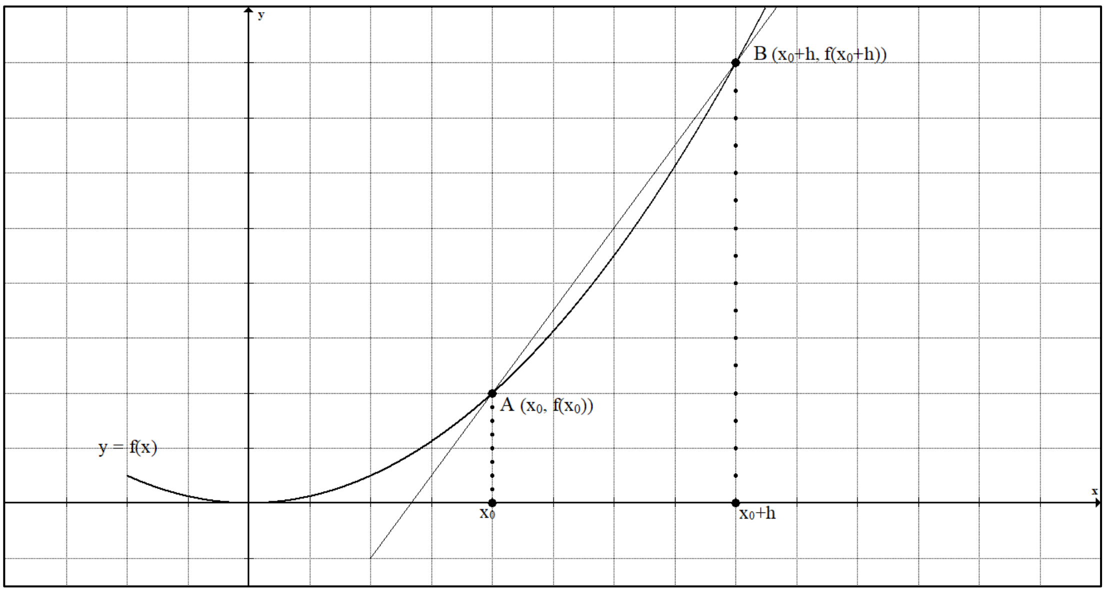
(ii) Apply the first principle of differentiation to determine the derivatives of given functions.

**References for further reading**

The lecture notes have been adopted from relevant topics from (Cowen et al., 1990; Kahenya, 2022; Stewart, 2012; Sullivan & Miranda, 2019).

**Introduction**

Consider the curve of the graph $y = f(x)$. The secant line from the point $(x_0, f(x_0))$ to the point $(x_0 + h, f(x_0 + h))$ is an approximation of the curve at the point $(x_0, f(x_0))$ as $h$ tends to zero.



The slope or gradient of the secant AB is given by;

$$\frac{f(x_0+h)-f(x_0)}{h} \ldots\ldots(i)$$

As $h$ approaches zero i.e., $h \to 0$, we get a better approximate of the curve.

Note that $h$ is a small change along the $x$ axis and we denote it as $\Delta x$.

Therefore, by definition; slope $= \frac{\text{change in y−coordinates}}{\text{change in x coordinates}}$

This implies $f(x_0 + \Delta x) - f(x_0)$ is the change in y coordinates and $(x_0 + \Delta x) - x_0$ is the change in x coordinates.

Therefore;

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{(x_0 + \Delta x) - x_0}$$

As $\Delta x \to 0$, point B will move along the curve and approach point A, such that the secant line AB approach the tangent line at point A.

**The slope or gradient of the tangent line is the same as the gradient of the curve at that point A.**

That is;

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{(x_0 + \Delta x) - x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

**Definition 1: Derivative of a function by first principle**

Given a function $f(x)$ then the function $f'(x)$ or $\frac{dy}{dx}$ is the derivative of $f(x)$ at point $x_0$ defined by;

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x}$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Alternatively;

$$\frac{dy}{dx} = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0}\right)$$

When the limit exists, we say that $f(x)$ is differentiable at point $x_0$.

**Definition 2: Derivative Notation**

Differentiation is the operation of finding the derivative function $f'(x)$ or $\frac{dy}{dx}$.

$\frac{dy}{dx}$ is called the Leibniz notation.

**Example 1:** Find the gradient of the tangent line to the curve $f(x) = x^2$ at point $x = 3$.

By definition

$\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$, Our $x_0 = 3$

Thus;

$$\lim_{\Delta x \to 0} \frac{f(3+\Delta x) - f(3)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(3+\Delta x)^2 - 9}{\Delta x} = \lim_{\Delta x \to 0} \frac{9 + 6\Delta x + (\Delta x)^2 - 9}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(6+\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} (6 + \Delta x) = 6$$

That is, the gradient of the tangent to the curve $f(x) = x^2$ at $x = 3$ is 6. (Alternatively, the gradient of the curve at point $x = 3$ is 6).

**Example 2:** Use the definition $f'(x) = \lim_{\delta x \to 0} \left(\frac{f(x_0+\delta x)-f(x_0)}{\delta x}\right)$ to find the derivative of;

$$f(x) = 2x^2 - 7$$

**Solution:**

$$f'(x) = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x_o + \delta x) - f(x_o)}{\delta x} = \lim_{\delta x \to o} \frac{(2(x_0 + \delta x)^2 - 7) - (2x_0^2 - 7)}{\delta x}$$

$$= \lim_{\delta x \to 0} \left(\frac{2x_0^2 + 4x_0\delta x + 2(\delta x)^2 - 7 - 2x_0^2 + 7}{\delta x}\right)$$

$$= \lim_{\delta x \to 0} \left(\frac{4x_0\delta x + 2(\delta x)^2}{\delta x}\right)$$

$$= \lim_{\delta x \to 0} \left(\frac{\delta x(4x_0 + 2\delta x)}{\delta x}\right) = \lim_{\delta x \to 0} (4x_0 + 2\delta x) = 4x_0$$

Clearly;

$$\frac{dy}{dx} = 4x$$

**Example 3:** Use the definition $f'(x) = \lim_{\delta x \to 0} \left(\frac{f(x_0+\delta x)-f(x_0)}{\delta x}\right)$ to find the derivative of $f(x) = \sin x$

**Solution:**

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \left(\frac{f(x_0 + \delta x) - f(x_0)}{\delta x}\right) = \lim_{\delta x \to 0} \left(\frac{\sin(x_0 + \delta x) - \sin x_0}{\delta x}\right)$$

$$\lim_{\delta x \to 0} \left(\frac{\sin(x_0 + \delta x) - \sin x_0}{\delta x}\right) = \lim_{\delta x \to 0} \frac{\sin x_0 \cos \delta x + \sin \delta x \cos x_0 - \sin x_0}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin x_0 (\cos \delta x - 1) + \sin \delta x \cos x_0}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin x_0(\cos \delta x - 1)}{\delta x} + \lim_{\delta x \to 0} \frac{\sin \delta x \cos x_0}{\delta x}$$

$$= \sin x_0 \lim_{\delta x \to 0} \frac{\cos(\delta x - 1)}{\delta x} + \cos x_0 \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} \cdots \text{(i)}$$

$$\text{Now } \lim_{\delta x \to 0} \frac{\cos(\delta x - 1)}{\delta x} = 0 \text{ and } \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} = 1$$

Hence (i) becomes;

$$\frac{dy}{dx}(\sin x) = \cos x$$

**Definition 4: Tangent line**

Suppose that the function $f(x)$ is differentiable at point $a$. Then the tangent to the graph of $y = f(x)$ at say point $(x_0, f(x_0))$ is the line through this point of slope $f'(x_0)$. The equation of the tangent line in point-slope form is;

$$y - f(x_0) = f'(x_0)(x - x_0)$$

**Example 1:** Determine the equation of the tangent line to the graph of the function;

$f(x) = x^2 + 2x + 1$ at $x = 1$.

**Solution:** We first compute $f'(1)$ i.e.

$$f'(1) = \lim_{x \to 1}\left(\frac{f(x) - f(1)}{x - 1}\right) = \lim_{x \to 1}\left(\frac{x^2 + 2x + 1 - 4}{x - 1}\right)$$

$$\lim_{x \to 1}\left(\frac{x^2 + 2x - 3}{x - 1}\right) = \frac{0}{0} - \text{indeterminate}$$

We need to simplify the numerator to get;

$$f'(1) = \lim_{x \to 1}\left(\frac{x^2 + 2x - 3}{x - 1}\right) = \lim_{x \to 1}\left(\frac{(x + 3)(x - 1)}{x - 1}\right) = \lim_{x \to 1}(x + 3) = 4$$

Now we use

$$y - f(x_0) = f'(x_0)(x - x_0)$$

To get;

$$y - f(1) = f'(1)(x - 1)$$

$$y - 4 = 4(x - 1)$$

$$y = 4x$$

This is the equation of the tangent to the graph of $f(x) = x^2 + 2x + 1$ at $x = 1$.

**Definition 5: Continuity and differentiation**

A function $f(x)$ is said to be differentiable at a point $x_0$ if the function is defined at least on some open interval, I containing the point $x_0$ and $f'(x)$exists.

**Proof:** Assume f is differentiable at point $x_0$, then $\lim_{x \to x_0} f(x) = l$ if and only if $\lim_{h \to 0} f(x_0 + h) = l$

$f(x)$ will be continuous at $x_0$ if $\lim_{\Delta x \to 0} f(x + \Delta x) = f(x)$

Since the limit of the product is the product of the limits

$$\lim_{\Delta x \to 0} [f(x + \Delta x) - f(x)] = \lim_{\Delta x \to 0} [\frac{f(x + \Delta x) - f(x)}{\Delta x} \Delta x]$$

$$= [\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}] \lim_{\Delta x \to 0} \Delta x$$

$$= f'(x) \times 0 = 0$$

Therefore, since the limit of a sum is the sum of the limits, this implies;

$$\lim_{\Delta x \to 0} f(x + \Delta x) = \lim_{\Delta x \to 0} [f(x + \Delta x) - f(x) + f(x)]$$

$$= \lim_{\Delta x \to 0} [f(x + \Delta x) - f(x)] + \lim_{\Delta x \to 0} f(x) = 0 + f(x) = f(x)$$

Alternatively, if $f(x)$ is differentiable at $x_0$, then $f(x)$ is continuous at $x_0$.

**Definition 6:** A function $f(x)$ is differentiable at $x_0$ with its derivative $f'(x)$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that;

$$|x - x_0| < \delta \Rightarrow \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \varepsilon$$

**Example 1:** Show that the derivative of $f(x) = 3x^2$ at $x = 1$ is 6.

**Proof:** Our $x_0 = 1, f(x) = 3x^2, f(x_0) = 3, f'(x_0) = 0$

$$|x - x_0| < \delta \Rightarrow \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \varepsilon$$

$$|x - 1| < \delta \Rightarrow \left| \frac{3x^2 - 3}{x - 1} - 0 \right| < \varepsilon$$

Working with the RHS to get;

$$\left| \frac{3x^2 - 3}{x - 1} - 0 \right| < \varepsilon \sim \left| \frac{3(x^2 - 1)}{x - 1} \right| < \varepsilon$$

$$\sim \left| \frac{3(x - 1)(x + 1)}{x - 1} \right| < \varepsilon$$

$$\sim \frac{3|x - 1||x + 1|}{|x - 1|} < \varepsilon$$

$$\sim 3|x - 1||x + 1| < \varepsilon |x - 1|$$

$$\sim|\mathrm{x}-1| < \frac{\varepsilon}{3}\cdot\frac{|x-1|}{|x+1|}$$

Let $\delta = \frac{\varepsilon}{3}\cdot\frac{|x-1|}{|x+1|}$

We can let $|x-1| < 1 \cdots \text{(i)} \Rightarrow 0 < \mathrm{x} < 2$

Hence, $1 < x+1 < 3$

$$|\mathrm{x}-1| < \frac{\varepsilon}{3}\frac{|x-1|}{|x+1|} < \frac{\varepsilon}{9}\cdots \text{(ii)}$$

$$\Rightarrow \delta = \min\{1, \frac{\varepsilon}{9}\}$$

Indeed, the function is differentiable at this point.

**Application of first principle of differentiation**

This will help build on techniques of differentiation, rate of change and related changes in the coming lectures.

**Exercise**

1) Use the definition $f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$ to find the derivative of the following functions

   a) $f(x) = 5x^2 + 2x + 1$

   b) $f(x) = x^2 + 7x - 3$

   c) $f(x) = 2x^3 - 7x$

   d) $f(x) = \frac{5}{x}$

2) Find the gradient of the tangent line to the curves at the given points

   a) $f(x) = x - 3x^2; x = 2$

   b) $f(x) = 2x^3 + 1; x = 1$

   c) $f(x) = x^2 - 5x + 3; x = 2$

3) Determine the equation of the tangent to the following graphs at the given point.

   a) $g(x) = 3x^2 + 1, x = 1$

   b) $g(x) = 5 - 2x^2, x = -2$

   c) $h(x) = x^3 + 2x + 3, x = 0$

   d) $h(x) = x^3 - x + 1, x = 3$

4) Prove that $f(x) = |x|$ is not differentiable at point $x = 0$.

**References**

Cowen, R. ., Were, J. ., & Vaz, P. . (1990). *An Introduction to Calculus*. Nairobi University Press.

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Sullivan, M., & Miranda, K. (2019). *Calculus: Early Transcendentals* (second). W.H. Freeman and Company.